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Radiative Equilibrium: The Relation between the Spectral Energy Curve of a Star and the Law of Darkening of the Disc towards the Limb, with Special Reference to the Effects of Scattering and the Solar Spectrum

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VI. *Radiative Equilibrium: the Relation between the Spectral Energy Curve of a Star and the Law of Darkening of the Disc towards the Limb, with Special Reference to the Effects of Scattering and the Solar Spectrum.**

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Contents.

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§1. *Introduction.*

An examination from a theoretical standpoint of the form of the law of darkening of a stellar disc towards the limb needs no apology at the present time. A knowledge of this law is required in two astronomical studies of the first importance: one is the deduction of the orbits and densities of eclipsing binary stars from observations of the light curves, the other is the deduction of the angular diameters of stars from interferometer measurements. In both cases some assumption has to be made as to the distribution of intensity over the disc before the solution becomes precise. Now the sun is the only star for which the intensity-distribution is at present known in any detail; the observations of Abbot, Fowle and Aldrich, as well as those of earlier investigators, have determined the law of darkening both for the integrated radiation and for the separate wave-lengths. As regards other stars, indirect evidence is indeed provided as to the *existence* of darkening at the limb by the results for eclipsing variables, since in most cases the darkened solution gives a better agreement between the observed and computed light-curves,* but as yet numerical precision as to the amount of darkening is hardly to be expected. The interferometer method of measuring angular diameters is theoretically capable of determining the light distribution also from the position of the second minimum of visibility of the fringes†; but here again the realisation of this possibility is a matter for the future. In the absence, therefore, of direct observations, it would seem worth while to investigate the types of darkening predicted by theory, for stars of different temperatures and of different atmospheric constitutions, on suitable assumptions; and to examine also the converse problem, namely, that of the deductions it is possible to make as to the state of the star if its law of darkening is given.

It seems the more desirable to elucidate the principles underlying the existence of darkening, since certain misconceptions appear to exist on the subject.

For example, one interesting consequence of the theory of radiative equilibrium is that (on this hypothesis) the limb-centre ratio for any given wave-length increases with increasing temperature. Hence, in the case of the sun, an increasing value of the solar constant should be accompanied with decreasing contrast, in any given wave-length, between the brightness of the sun's centre and limb. Now some evidence has been found of the occurrence of long-period variations in the solar constant, and of correlation of these with changes in the limb-centre contrast,‡ increasing solar constant going with increased contrast; and in the theory put forward to account for this it is stated that "it is clear that if the solar temperature was zero the contrast of brightness would be zero, and the higher the temperature the higher the contrast and the higher

* Shapley, 'Orbits of Eclipsing Binaries,' p. 106, Princeton, 1915.

† Michelson and Pease, 'Astrophys. Journ.,' 53, p. 249 (1921).

‡ Abbot, Fowle and Aldrich, 'Misc. Smithson. Coll.,' 66, No. 5 (1917).

the solar constant of radiation." It is true that at the zero of temperature the intensity is the same at the centre and the limb, but the contrast ratio takes the indeterminate form 0/0, which need by no means be zero. The theoretical law is as given above, and thus simple increase of temperature does not explain the observations. A further consequence of the theory is, that the darkening is independent of the absolute transparency of the material composing the outer layers; this renders difficult the explanation offered of the apparent correlation of the day-to-day changes of the solar constant with the limb-centre contrast (a correlation which is in the opposite direction to that of the long-period changes), since it is suggested that changes in transparency are responsible. Here an explanation on the basis of temperature would be adequate. Similar difficulties occur with the integrated radiation, the darkening in which (at least for a gray body) should be an absolute constant, independent of both temperature and transparency.

Again STEBBINS,* discussing the light-curve of the eclipsing variable 1 H. Cassiopeiæ, estimates the degree of darkening to be expected in light of wave-length 4560A, assuming the star to have the same degree of darkening as the sun, and using the observed darkening for the sun in this wave-length. But this ignores the question of surface temperature. If we consider a hypothetical series of stars having the same general thermal structure, but different temperatures, the darkening in integrated light should be the same for all, but the darkening in a particular wave-length should decrease as the temperature increases, the darkening in wave-length λ being a function, not of λ alone, but of $\lambda/\lambda_{\max}$. An application of WIEN's law shows that the darkening in wave-length λ_1 in a star of temperature T_1 should follow the same law as that for wave-length λ_2 in a star of temperature T_2, where $\lambda_1 T_1 = \lambda_2 T_2$. Now 1 H. Cass. is a star of type B3, with a temperature, say, twice that of the sun. Consequently the darkening in $\lambda 4560$ is to be expected to follow the same law as that for $\lambda 9120$ in the sun; $\lambda 9120$ shows in the sun considerably less darkening than $\lambda 4560$, according to ABBOT the limb-intensity in $\lambda 4560$ being about $0 \cdot 25$ of the central and that in $\lambda 9120$ about $0 \cdot 55$ of the central. It should be mentioned that the actual darkening found by STEBBINS is much less even than this; in fact, he found no evidence of darkening at all. But this could equally well be interpreted to mean that the star had an exceptionally high temperature.

It does not appear to have been previously considered that there should be an intimate relation between the law of darkening of a star and the distribution of energy in its continuous spectrum. Causes such as varying absorption in different parts of the spectrum (departure from gray body properties) and molecular scattering, which distort the spectral energy curve, should also leave their mark on the law of darkening. It is one of the objects of this paper to examine these relationships, and to show how the observed darkening is theoretically capable of affording criteria as to the physical nature of the distorting causes. In particular, on certain simple assumptions, it will

* 'Astrophys. Journ.,' 54, p. 91, (1921).

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appear that there should be a sharp distinction between the effects of scattering and of absorption in modifying the law of darkening; extensive scattering without absorption implies the same law of darkening in all wave-lengths, the coefficient of darkening being an absolute constant. It is not permissible, therefore, to invoke scattering to account for a depression of the energy spectrum unless the darkening shows this characteristic.

To illustrate the general theory one naturally turns to the sun, the energy curve of which departs considerably from that of a black body. It is disappointing to have to record that on the whole the sun shows no confirmation. There are indeed certain minor anomalies in the observed darkening for the sun which can be partly accounted for; for example, the much higher temperatures deduced by other writers from the observed darkening than those indicated on other grounds. But the striking point that emerges is the surprisingly good agreement between the observed darkening and the theoretical darkening for a black body in radiative equilibrium—surprising, that is, in view of the energy curve. The usual explanations offered of the form of the energy curve imply quite definite consequential alterations in the darkening, alterations which are not observed. Neither scattering nor absorption seem competent to account for the form of the energy curve, if the theory is correct. It is difficult to be sure that some combination of hypotheses, or the insertion of more violent ones, will not afford an explanation in terms of scattering and absorption; but none is suggested that commends itself, and the phenomenon is certainly more complicated than it is usually taken to be.

In the investigation which follows, the question of absorption is first discussed, carrying on the work of an earlier paper; then purely scattering atmospheres are considered, both in their effect on the darkening and their effect on the continuous spectrum; lastly, combinations of scattering atmospheres, in front of or mixed with absorbing atmospheres, are treated.

§2. *The Law of Darkening for Gray Absorbing Matter in Radiative Equilibrium.*

Consider a mass of material stratified in parallel planes and possessing a plane outer boundary. Let the coefficient of absorption, k, be independent of wave-length (*i.e.* suppose the material to be gray); and assume that there is no scattering. Let ρ be the density at any point whose distance from the boundary is x. Then the optical thickness τ between the boundary and any point x is given by

$$\tau = \int_0^x k\rho\, dx\,;$$

τ will sometimes be spoken of as the optical depth of x, or, more briefly, we shall speak of "the point τ." Let T be the temperature at the point τ, $B_\lambda(\tau)$ the intensity of black body radiation, of wave-length λ, for the temperature T of the material at τ; further, let B (τ) be the intensity of integrated black body radiation for the temperature at τ. Thus, if σ is STEFAN'S constant,

$$B(\tau) = \int_0^\infty B_\lambda(\tau)\, d\lambda = \sigma T^4/\pi.$$

Let I_λ (τ, θ) be the intensity of radiation, of wave-length λ, actually traversing the material at the point τ in a direction making an angle θ with the outward normal to the planes of stratification. When $\frac{1}{2}\pi < \theta \leqslant \pi$, it will sometimes be convenient to write

$$\pi - \theta = \psi, \qquad I_\lambda(\tau, \theta) = I'_\lambda(\tau, \psi). \qquad (0 \leqslant \psi < \tfrac{1}{2}\pi).$$

For the total radiation of all wave-lengths the suffix λ is omitted, as above. At the boundary the intensity emergent in direction θ is I_λ (0, θ); and the inward intensity I'_λ (0, ψ) is zero.

At the boundary of a star the curvature is so small that over any limited region the material may be supposed stratified in parallel planes, in the way just considered. In this case I_λ (0, θ) is the observed intensity at a point on the disc whose angular distance from the centre of the disc, with reference to the centre of the star, is θ; the intensity at the limb is, of course, I_λ (0, $\frac{1}{2}\pi$).

It has been shown in an earlier paper* that for a star in which the foregoing conditions are satisfied, the emergent intensity is given in terms of the temperature distribution by the relation

$$I_\lambda(0, \theta) = \int_0^\infty B_\lambda(\tau \cos\theta)\, e^{-\tau}\, d\tau. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This formula is quite general; it applies to any temperature distribution, and it remains true whatever function k may be of the temperature or density, provided it is independent of λ. If now the temperature distribution is supposed to be determined by the condition of radiative equilibrium, it has been shown† that a satisfactory approximation to the function B (τ) is given by

$$B(\tau) = \tfrac{1}{2}F(1 + \tfrac{3}{2}\tau), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where πF is the constant net outward flux of radiation at any point τ, per unit area, characterising the state of radiative equilibrium. If T_1 is the effective temperature of the star, estimated from the total radiation and STEFAN'S law, then

$$\pi F = \sigma T_1^{\,4};$$

further, the boundary temperature is given by

$$B(0) = \tfrac{1}{2}F, \qquad \text{or} \qquad T_0^{\,4} = \tfrac{1}{2}T_1^{\,4},$$

(SCHWARZSCHILD'S formula), and that at any other point by

$$T^4 = T_0^{\,4}(1 + \tfrac{3}{2}\tau). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2\text{A})$$

* 'Monthly Notices, R.A.S.,' 81, p. 382, 1921 (referred to henceforth as Paper 2).

† 'Monthly Notices, R.A.S.,' 81, p. 361, 1921 (referred to henceforth as Paper 1).

Adopting now PLANCK's formula,

$$B_\lambda = \frac{2hc^2\lambda^{-5}}{e^{hc/\lambda RT} - 1},$$

and substituting for T, equation (1) becomes

$$I_\lambda(0, \theta) = 2hc^2\lambda^{-5}\int_0^\infty \frac{e^{-\tau}\,d\tau}{e^{\alpha(1+\frac{3}{2}\tau\cos\theta)^{-\frac{1}{4}}} - 1},$$

where we have written α for $hc/\lambda RT_0$. It is more convenient to re-write this formula thus—

$$I_\lambda(0, \theta) = KT_0^5\alpha^5\int_0^\infty \frac{e^{-\tau}\,d\tau}{e^{\alpha(1+\frac{3}{2}\tau\cos\theta)^{-\frac{1}{4}}} - 1}, \qquad (3)$$

where K stands for $2hc^2(R/hc)^5$.

It is necessary to consider the behaviour of the function

$$f(\alpha, p) = \alpha^5\int_0^\infty \frac{e^{-\tau}\,d\tau}{e^{\alpha(1+p\tau)^{-\frac{1}{4}}} - 1}. \qquad (3\text{A})$$

For p constant it has the general character of a spectral energy curve, having one maximum and vanishing for very large and very small values of α. When it is plotted against $1/\alpha$, $(\propto \lambda)$, to give a "normal" spectrum, the area enclosed between it and the axis is given by

$$\int_0^\infty f(\alpha, p)\, d\left(\frac{1}{\alpha}\right) = \frac{\pi^4}{15}(1+p).$$

The function is an increasing function of p; as p increases the peak becomes more pronounced and travels off in the direction of α increasing (shorter wave-lengths). Methods of computing $f(\alpha, p)$ have been considered elsewhere,* and an asymptotic formula† has been given for α large.

The "law of darkening" is the formula expressing the ratio of the intensity at any point of the disc to the intensity at the centre. From (3) the law of darkening in wave-length λ is given by

$$\frac{I_\lambda(0, \theta)}{I_\lambda(0, 0)} = \frac{f(hc/\lambda RT_0, \frac{3}{2}\cos\theta)}{f(hc/\lambda RT_0, \frac{3}{2})}. \qquad (4)$$

The first point to notice is that the law of darkening in wave-length λ depends only on the product λT_0, and not on λ and T_0 separately. Thus, if the law has been evaluated for all wave-lengths for a particular T_0, the law for a star of any other effective temperature is also known and may be determined as mentioned in § 1. It is found that $f(\alpha, \frac{3}{2}\cos\theta)/f(\alpha, \frac{3}{2})$ decreases as α increases; thus the darkening becomes more

* Paper 2, Appendix.

† See equation (61) below.

pronounced as λT_0 decreases; the darkening for given T_0 becomes more pronounced as λ decreases, and for given λ becomes more pronounced as T_0 decreases. This is merely an expression of the fact that the intensity of radiation in any wave-length increases more rapidly with the temperature the shorter the wave-length. Further, the larger is λT_0, the smaller is the rate of change of the darkening with respect to λT_0. In fact, for large values of λT_0 the law of darkening tends to a definite limiting form which is easily calculated. For when α is small (*i.e.* λT_0 large),

$$f(\alpha, p) = \alpha^4 \int_0^\infty (1+p\tau)^{\frac{1}{4}} e^{-\tau} \, d\tau,$$

approximately, so that

$$\lim_{\lambda T_0 \to \infty} \frac{I_\lambda(0, \theta)}{I_\lambda(0, 0)} = \frac{\int_0^\infty (1+\frac{3}{2}\tau \cos\theta)^{\frac{1}{4}} e^{-\tau} \, d\tau}{\int_0^\infty (1+\frac{3}{2}\tau)^{\frac{1}{4}} e^{-\tau} \, d\tau} \qquad (5)$$

This may also be expressed in the statement that for all λ's for which $\lambda/\lambda_{\max}$ is sufficiently great, the law of darkening is the same. To see what this limiting law is, we will evaluate the limiting value of the limit-centre ratio. Numerical calculation shows that

$$\int_0^\infty (1+\tfrac{3}{2}\tau)^{\frac{1}{4}} e^{-\tau} \, d\tau = 1 \cdot 2242\,;$$

the limiting value of the limb-centre ratio is the reciprocal of this, namely, $0 \cdot 817$. Had we taken for the temperature distribution the alternative approximation for radiative equilibrium (see Paper 1),

$$B = \tfrac{3}{8} F (1+2\tau), \qquad T_0^4 = \tfrac{3}{8} T_1^4, \qquad (6)$$

we should have had to replace (5) by

$$\lim_{\lambda T_0 \to \infty} \frac{I_\lambda(0, \theta)}{I_\lambda(0, 0)} = \frac{\int_0^\infty (1+2\tau \cos\theta)^{+\frac{1}{4}} e^{-\tau} \, d\tau}{\int_0^\infty (1+2\tau)^{\frac{1}{4}} e^{-\tau} \, d\tau},$$

and the limit of the limb-centre ratio would have been $1/1 \cdot 2728 = 0 \cdot 786$. The true temperature distribution lies between the two. Hence, we have the result: as λT_0 (or $\lambda/\lambda_{\max}$) increases, the ratio of the intensity at the limb to the intensity at the centre steadily increases, but never exceeds the value $0 \cdot 8$ approximately. In particular this is the darkening to which the visible spectrum will tend as $\lambda_{\max}$, by increase of temperature, moves into the ultra-violet. It must be remembered that direct observations, if they could be made, would apparently indicate much less darkening than this; for they could never be made exactly at the limb, and even for a point 95 per cent. of the radius from the centre the value of $\cos\theta$ is $0 \cdot 312$, considerably different from the limb value zero.

On the other hand, for small values of λT_0 (or λ/λ_{max}) the darkening increases, the limit as $\lambda T_0 \rightarrow 0$ being complete darkening at the limb. This can be shown by means of the asymptotic formula for α large.

To be clearly separated from these results is the fact that for the integrated radiation the darkening is independent of temperature. We find, in fact, that with the approximation (2),

$$\int_0^\infty I_\lambda(0, \theta)\, d\lambda = \tfrac{1}{2}F\left(1+\tfrac{3}{2}\cos\theta\right),$$

so that, as the temperature increases, the decreased darkening in the longer wave-lengths is exactly compensated by the increased energy thrown into the shorter wave-lengths which show the greater darkening.

The general inference is that on strict radiative equilibrium, with no selective absorption and no scattering, blue stars should show decreased contrast in any given wave-length, as compared with red stars, but should show the same darkening in integrated radiation.

§3. *Application to the Sun.*

Adopting Abbot's value of the solar constant 1·932 and Coblentz' value of Stefan's constant,* $\sigma = 5\cdot70 \times 10^{-5}$ ergs. sec.$^{-1}$ cm.$^{-2}$, or $8\cdot19 \times 10^{-11}$ cals. min.$^{-1}$ cm.$^{-2}$, the effective temperature T_1 for the sun comes out as 5740° absolute, giving for the boundary temperature $T_0 = 2^{-\frac{1}{4}} \times 5740 = 4830°$. Using now the value $hc/R = 1\cdot4325$ (which is consistent with the above value of σ), the quantity α may be calculated for any given λ; hence the law of darkening may be calculated from (4), for any given λ, provided the function $f(\alpha, p)$ is known. Appendix I (p. 246) gives tables of $f(\alpha, p)$ for various values of α and for $p = 0, \tfrac{1}{2}, 1, 1\tfrac{1}{2}, 2$; from these, curves were drawn from which, for these p's, the values of f could be read off for any required value of α; these values were then re-plotted against p, and points read off corresponding to the odd values of p given by $p = \tfrac{3}{2}\cos\theta$. In this way the theoretical value of the darkening ratio was computed for the separate wave-lengths and separate places on the sun's disc given by Abbot, and the results compared with Abbot's observations.† In Table I the first row in each compartment gives the observed value; the row marked (i) gives the computed value according to (4). (Rows (ii), (iii), (iv) are explained in §§ 4, 6, 9, respectively.)

It is at once apparent that the general run of the agreement is very good, but that the calculated intensity ratios (except for λ4330) are systematically slightly less than the observed ratios; the simple theory predicts rather more darkening than is observed.

* This value is larger than Kurlbaum's value, $5\cdot45 \times 10^{-5}$, which has usually been employed in this connection; hence our smaller value of T_1.

† 'Annals Astrophys. Obs. Smithson. Instit.,' 3, p. 157 (1913.)

TABLE I.—Relative Darkening over the Sun's Disc.

λ (A.U.)	Sin θ	0·000	0·200	0·400	0·550	0·650	0·750	0·825	0·875	0·920	0·950
	Cos θ	1·000	0·980	0·916	0·835	0·760	0·661	0·565	0·484	0·392	0·312
3230	Observed	1·000	0·960	0·897	0·835	0·775	0·690	0·600	0·530	0·452	0·382
	(i)		0·976	0·908	0·818	0·737	0·637	0·541	0·469	0·393	0·332
	(ii)		0·980	0·911	0·823	0·746	0·648	0·560	0·491	0·418	0·360
	(iii)		0·980	0·921	0·855	0·792	0·713	0·642	0·585	0·521	0·469
	(iv)		0·976	0·912	0·829	0·753	0·659	0·573	0·507	0·437	0·380
3860	Observed	1·000	0·980	0·926	0·856	0·792	0·710	0·633	0·554	0·483	0·418
	(i)		0·982	0·925	0·850	0·782	0·694	0·611	0·543	0·470	0·408
	(ii)		0·983	0·927	0·855	0·788	0·705	0·625	0·562	0·491	0·433
	(iii)		0·982	0·925	0·851	0·783	0·698	0·617	0·551	0·480	0·420
	(iv)		0·983	0·932	0·865	0·804	0·722	0·645	0·582	0·514	0·455
4330	Observed	1·000	0·978	0·927	0·866	0·806	0·729	0·647	0·583	0·510	0·450
	(i)		0·983	0·935	0·870	0·811	0·732	0·655	0·591	0·518	0·457
	(ii)		0·985	0·937	0·875	0·817	0·741	0·667	0·605	0·536	0·481
	(iii)		0·985	0·933	0·865	0·803	0·722	0·642	0·575	0·499	0·433
	(iv)		0·985	0·940	0·881	0·827	0·755	0·685	0·627	0·563	0·507
4560	Observed	1·000	0·986	0·942	0·885	0·831	0·756	0·681	0·616	0·538	0·471
	(i)		0·984	0·938	0·875	0·818	0·744	0·671	0·610	0·540	0·480
	(ii)		0·986	0·940	0·880	0·826	0·754	0·685	0·626	0·560	0·504
	(iii)		0·983	0·932	0·866	0·805	0·727	0·648	0·581	0·504	0·440
	(iv)		0·986	0·944	0·890	0·838	0·771	0·706	0·651	0·589	0·533
4810	Observed	1·000	0·987	0·944	0·891	0·840	0·771	0·701	0·638	0·566	0·499
	(i)		0·985	0·942	0·883	0·828	0·757	0·687	0·629	0·563	0·507
	(ii)		0·987	0·944	0·887	0·836	0·768	0·702	0·648	0·584	0·529
	(iii)		0·985	0·935	0·872	0·813	0·734	0·660	0·595	0·526	0·459
	(iv)		0·987	0·946	0·896	0·848	0·785	0·723	0·670	0·611	0·557
5010	Observed	1·000	0·985	0·945	0·895	0·845	0·777	0·711	0·650	0·583	0·517
	(i)		0·986	0·944	0·887	0·834	0·766	0·699	0·644	0·580	0·523
	(ii)		0·988	0·946	0·892	0·842	0·778	0·714	0·662	0·600	0·548
	(iii)		0·985	0·937	0·877	0·820	0·745	0·672	0·610	0·540	0·478
	(iv)		0·988	0·948	0·898	0·851	0·789	0·729	0·676	0·619	0·566
5340	Observed	1·000	0·987	0·950	0·902	0·856	0·792	0·728	0·672	0·605	0·548
	(i)		0·988	0·946	0·893	0·545	0·780	0·716	0·664	0·601	0·548
	(ii)		0·988	0·950	0·900	0·853	0·792	0·732	0·682	0·623	0·572
	(iii)		0·988	0·943	0·885	0·834	0·763	0·692	0·632	0·564	0·504
	(iv)		0·989	0·955	0·908	0·865	0·807	0·750	0·702	0·644	0·593
6040	Observed	1·000	0·989	0·957	0·913	0·872	0·816	0·761	0·710	0·648	0·593
	(i)		0·989	0·952	0·905	0·861	0·804	0·745	0·696	0·640	0·589
	(ii)		0·990	0·956	0·913	0·871	0·816	0·762	0·716	0·662	0·613
	(iii)		0·989	0·950	0·901	0·854	0·788	0·724	0·669	0·606	0·550
	(iv)		0·990	0·958	0·915	0·876	0·823	0·772	0·727	0·675	0·628
6700	Observed	1·000	0·991	0·961	0·924	0·887	0·838	0·786	0·740	0·680	0·629
	(i)		0·990	0·958	0·916	0·876	0·824	0·770	0·724	0·669	0·618
	(ii)		0·991	0·960	0·920	0·883	0·832	0·781	0·737	0·685	0·640
	(iii)		0·990	0·954	0·908	0·865	0·808	0·750	0·701	0·642	0·587
	(iv)		0·991	0·961	0·923	0·887	0·838	0·790	0·747	0·699	0·655

TABLE I. (continued).

λ (A.U.)	Sin θ	0·000	0·200	0·400	0·550	0·650	0·750	0·825	3·875	0·920	0·950
	Cos θ	1·000	0·980	0·916	0·835	0·760	0·661	0·565	0·484	0·392	0·312
6990	Observed	1·000	0·990	0·963	0·926	0·890	0·841	0·792	0·748	0·691	0·637
	(i)		0·991	0·960	0·920	0·883	0·830	0·778	0·733	0·680	0·632
	(ii)		0·992	0·963	0·923	0·887	0·838	0·788	0·746	0·695	0·651
	(iii)		0·989	0·957	0·913	0·873	0·818	0·763	0·683	0·655	0·603
	(iv)		0·991	0·963	0·925	0·890	0·842	0·795	0·756	0·707	0·664
8660	Observed	1·000	0·992	0·969	0·939	0·911	0·871	0·830	0·792	0·744	0·699
	(i)		0·993	0·967	0·936	0·903	0·860	0·813	0·775	0·727	0·685
	(ii)		0·993	0·968	0·935	0·904	0·863	0·820	0·782	0·738	0·699
	(iii)		0·992	0·969	0·936	0·903	0·862	0·820	0·781	0·738	0·697
	(iv)		0·992	0·969	0·939	0·908	0·868	0·828	0·790	0·750	0·714
10310	Observed	1·000	0·998	0·977	0·951	0·925	0·889	0·851	0·816	0·772	0·730
	(i)		0·994	0·970	0·945	0·917	0·875	0·837	0·798	0·756	0·716
	(ii)		0·994	0·972	0·943	0·916	0·877	0·840	0·805	0·763	0·730
	(iii)		0·993	0·975	0·945	0·920	0·884	0·847	0·816	0·780	0·744
	(iv)		0·994	0·975	0·951	0·925	0·891	0·855	0·827	0·790	0·758
12250	Observed	1·000	0·995	0·976	0·953	0·932	0·901	0·865	0·834	0·794	0·756
	(i)		0·994	0·973	0·945	0·920	0·886	0·848	0·816	0·778	0·741
	(ii)		0·995	0·976	0·950	0·930	0·894	0·850	0·830	0·791	0·757
	(iii)		0·992	0·970	0·945	0·921	0·885	0·852	0·818	0·780	0·741
	(iv)		0·995	0·980	0·955	0·933	0·901	0·865	0·833	0·800	0·763

Before discussing the matter further it is necessary to refer to the work of LINDBLAD,* who has made a similar comparison between the observed darkening and that calculated on radiative equilibrium. In principle, his method is identical with ours, but the details are very different. He begins by adopting the SCHWARZSCHILD approximation

$$T^4 = T_0^{\,4}(1+2\tau), \qquad T_0^{\,4} = \tfrac{1}{2}T_1^{\,4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which differs, it should be noted, from (6); it does not satisfy the condition of radiative equilibrium in the interior,† which requires $T_0^{\,4} = \frac{3}{8}\,T_1^{\,4}$. He then derives a formula similar to (4), with the $\frac{3}{2}$ replaced by 2, and uses this to determine, for each λ, the best value of T_0 to give the observed darkening. If the theory fitted perfectly, the resulting values of T_0 should all be the same. The mean value of T_0 comes out as 5430°. He

* 'Uppsala Universitets Årsskrift,' 1920, p. 1.

† *Note added August* 1, 1922.—This is perhaps a little obscurely expressed. If the relation $T^4 = T_0^{\,4}(1+2\tau)$ is adopted as an approximation to the temperature distribution in radiative equilibrium, then it is found that for the condition of radiative equilibrium to be satisfied in the far interior we must have $T_0^{\,4} = \frac{3}{8}T_1^{\,4}$, where $\sigma T_1^{\,4}$ is the net outward flux and accordingly T_1 is the effective temperature; this is approximation (6) above. If the more accurate approximation $T^4 = T_0^{\,4}(1+\frac{3}{2}\tau)$ is adopted, then we require $T_0^{\,4} = \frac{1}{2}T_1^{\,4}$; this is approximation (2A) above. Equations (7) are thus, strictly speaking, inconsistent. Any linear function of τ which is an approximation to T^4 must be of the form $T^4 = T_0^{\,4} + \frac{3}{4}T_1^{\,4}\tau$. (See Paper 1.)

then deduces the effective temperature T_1; not, however, from (7), but by multiplying T_0 by $3^{\frac{1}{4}}$ to give the effective temperature at the centre of the disc, and then reducing this in accordance with the observed value of the ratio of the mean brightness of the disc to the central brightness. (This is practically equivalent to taking $T_0^4 = \frac{3}{7} T_1^4$, in approximate agreement with the theoretically more correct form of the SCHWARZSCHILD formula.) The result is $T_1 = 6720°$.

This value is very considerably higher than that (5740°) deduced from the solar constant, even higher than that deduced from the observed position of $\lambda_{\max}$ (6130°), or from the observed intensity at $\lambda_{\max}$ (6120°).* Now it is interesting to examine why this method leads to so high a temperature. In the first place, our calculations have shown that the sun is less darkened, in any given λ, than the simple theory requires. But from §2 decreased darkening in any given λ is given by a higher temperature; consequently, if we use the observed darkening to determine the temperature, the latter will come out higher. In the second place, our calculations based on (2) require the coefficient of darkening, in integrated light, to be $\frac{3}{5}$, whilst formula (7) requires it to be $\frac{2}{3}$. It thus demands still more darkening, and requires a still higher temperature to compensate it.

If the solar temperature can really be deduced in this way, and if the ascertained value has this high value, then we have an important contribution to solar physics. But this procedure of LINDBLAD'S violates the assumptions that were made at the beginning of it. The temperature distribution is calculated on the basis of radiative equilibrium, and the chief characteristic of radiative equilibrium is the existence of a constant net flux of radiation of amount σT_1^4; the magnitude of this flux determines the temperature gradient and is itself determined by the amount of heat leaving the surface—by the solar constant, in fact. But the value of T_1 determined by LINDBLAD'S method has no connection with this the original definition of T_1; it ignores the value of the solar constant, demanding indeed nearly twice the accepted value. If it be objected that although (7) was derived on radiative equilibrium it is really only used as a working form of specification of the temperature distribution, then it and the derived law of darkening sink to the level of an interpolation formula, and the quantities are robbed of their physical significance; further, it is in that case more logical to assume a formula

$$T^4 = a_0 + a_1\tau$$

and determine both constants to fit the observations. Again, it may be contended that the observed radiation is considerably less than that leaving the photosphere, owing to absorption and scattering, and that this would account for the bigger solar constant required. But, if so, what happens to the energy lost by absorption? If the outer layers are on the whole in a steady state, it must be radiated back to the photosphere;

* See Appendix II. LINDBLAD'S temperatures are based on $\sigma = 7\cdot67 \times 10^{-11}$ cal. min.$^{-1}$ cm.$^{-2}$, but this makes them only $1\cdot6$ per cent. or 100° higher than they otherwise would be.

but in that case the *net* flux corresponds to the observed solar constant, as before, and further the boundary condition employed in the deduction of (7), namely, incident radiation zero at the boundary, is violated. It is remarkable how freely effects are ascribed to absorption without the fate of the absorbed energy being taken into account. LINDBLAD, while invoking a scattering atmosphere to reduce the radiation corresponding to 6720° to that observed, does not examine whether this scattering atmosphere may not modify the law of darkening of the bright background.

It will appear that the observed darkening does not imply such a high temperature, although it does imply one somewhat higher than the effective temperature 5740°. The difficulty is in part to reconcile the low temperature given by the darkening with the high temperature indicated by portions of the continuous spectrum.

§4. *The Law of Darkening in Integrated Light.*

It was shown in Paper 1 that on strict radiative equilibrium the law of darkening corresponds to a mean coefficient of darkening* ($\bar{u}$) equal to about 0·61 or 0·62;

* *Note added August* 1, 1922.—For the convenience of the reader the following definitions are inserted:—

If the ratio of the intensity of radiation emergent at an angle θ with the normal to the intensity emergent normally is a linear function of cos θ, so that we can write

$$\frac{\mathrm{I}(0, \theta)}{\mathrm{I}(0, 0)} = 1 - u + u \cos \theta,$$

where u is some constant, then u is said to be the "coefficient of darkening." In terms of the central and limb intensities only, u is given by the relation

$$u = 1 - \frac{\mathrm{I}(0, \frac{1}{2}\pi)}{\mathrm{I}(0, 0)}.$$

If the intensity-ratio is not a linear function of cos θ, then we can define a "mean coefficient of darkening," $\bar{u}$, as such that the ratio of the mean intensity over the disc to the intensity at the centre is the same as if the disc had a coefficient of darkening equal to $\bar{u}$. The mean intensity over the disc, say I_m, is given by

$$\mathrm{I}_m = \frac{\int_0^{\frac{1}{2}\pi} \mathrm{I}(0, \theta) \cos \theta \sin \theta \, d\theta}{\int_0^{\frac{1}{2}\pi} \cos \theta \sin \theta \, d\theta} = 2 \int_0^{\frac{1}{2}\pi} \mathrm{I}(0, \theta) \cos \theta \sin \theta \, d\theta,$$

and comparing this with the equation giving the net emergent flux per unit area,

$$\pi \mathrm{F} = 2\pi \int_0^{\frac{1}{2}\pi} \mathrm{I}(0, \theta) \cos \theta \sin \theta \, d\theta,$$

we see that

$$\mathrm{I}_m = \mathrm{F}.$$

The mean coefficient of darkening $\bar{u}$ is now given by the relation

$$\frac{\mathrm{F}}{\mathrm{I}(0, 0)} = \frac{\int_0^{\frac{1}{2}\pi} (1 - \bar{u} + \bar{u} \cos \theta) \cos \theta \sin \theta \, d\theta}{\int_0^{\frac{1}{2}\pi} \cos \theta \sin \theta \, d\theta} = 1 - \tfrac{1}{3}\bar{u}.$$

Similar definitions hold for the intensities in the separate wave-lengths. (In the case of the sun the integrated intensity follows a strict cosine law much more closely than do the intensities in the separate wave-lengths; this is in accordance with theory.)

the value $u = 3/5$ gives a good approximation over most of the disc, save near the limb, where there is a more sudden fall in intensity. A careful re-integration of ABBOT's results for the sun for the separate wave-lengths, with suitable allowances for the ultra-violet and infra-red regions, now makes possible a more detailed comparison. This is shown in Table II. By further integration it is found that the ratio of mean

TABLE II.

Sin θ	0·00	0·40	0·55	0·65	0·75	0·825	0·875	0·92	0·95
Cos θ	1·0000	0·9165	0·8352	0·7599	0·6614	0·5651	0·4841	0·3919	0·3122
Observed ...	1·000	0·955	0·912	0·871	0·822	0·769	0·722	0·665	0·612
$u = \frac{3}{5}$	1·000	0·950	0·901	0·856	0·797	0·739	0·690	0·635	0·587
$u = 0{\cdot}54$ ...	1·000	0·955	0·911	0·870	0·817	0·765	0·721	0·672	0·629
$u = 0{\cdot}56$ ...	1·000	0·953	0·908	0·865	0·810	0·756	0·711	0·660	0·615

brightness over the disc to the central brightness is 0·8205, whence the mean coefficient $\bar{u}$ is given by

$$1-\tfrac{1}{3}\bar{u} = 0{\cdot}8205 \quad \text{or} \quad \bar{u} = 0{\cdot}54.$$

Near the limb the darkening is slightly more than this, corresponding perhaps to $u = 0{\cdot}56$. The darkening is thus considerably less than the theoretical value for radiative equilibrium. The correct explanation of this is probably that given by LINDBLAD, taking into account the observed form of the energy spectrum; the depression of this in the violet prevents the violet rays (which are the most darkened) from contributing their proper share to the total energy fall from the centre outwards. But this only makes it the more remarkable that, though the absolute values of the emergent radiation, $KT_0^5 f(\alpha, p)$, are so different from the observed, their ratios should agree so well with the observed ratios.

Let us examine the consequences of accepting the observed law of darkening in integrated light, ignoring the equilibrium of radiation. A coefficient of darkening u may be considered to imply a temperature distribution given by

$$T^4 = T_0^4\left(1+\frac{u}{1-u}\tau\right),$$

and hence a law of darkening in the separate λ's given by

$$I_\lambda(0, \theta) = KT_0^5 f\left(\alpha, \frac{u\cos\theta}{1-u}\right).$$

The rows marked (ii) in Table I. give the darkening calculated from this formula with $u = 0 \cdot 56$ and $T_0 = 4830°$ as before. The agreement with the observed values is quite good, except for the last column.* Thus, if a slight empirical change in the temperature distribution from that given by strict radiative equilibrium is made, emission and absorption on simple black body principles is competent to explain the intensity distribution. The exact significance of this result is difficult to state, and it is perhaps dangerous to press the agreement; also the difficulty raised above as to the form of the spectrum still remains; but the fact that the agreement results from the assumption of a coefficient of absorption independent of wave-length, combined with a use of the numerical form of PLANCK's law, should not be lost sight of.

It may at first sight seem a numerical error that for the last column the calculated values should almost all *exceed* the observed, when the integrated radiation has been made to agree, but it can be shown that this is quite possible when the form of the energy curve is taken into account. For if $i_\lambda(\theta)$ is the actual intensity, $I_\lambda(\theta)$ the calculated, we have to show that the condition

$$\frac{\int I_\lambda(\theta)\, d\lambda}{\int I_\lambda(0)\, d\lambda} = \frac{\int i_\lambda(\theta)\, d\lambda}{\int i_\lambda(0)\, d\lambda}$$

is compatible with the condition that for all λ

$$\frac{I_\lambda(\theta)}{I_\lambda(0)} > \frac{i_\lambda(\theta)}{i_\lambda(0)}.$$

Set

$$\frac{i_\lambda(0)}{\int i_\lambda(0)\, d\lambda} = w_\lambda, \qquad \frac{I_\lambda(0)}{\int I_\lambda(0)\, d\lambda} = W_\lambda.$$

Then, eliminating $i_\lambda(\theta)$, this requires that

$$\int I_\lambda(\theta)\, d\lambda < \int I_\lambda(\theta) \frac{w_\lambda}{W_\lambda} d\lambda,$$

which is possible if $w_\lambda > W_\lambda$ when I_λ is large, and $w_\lambda < W_\lambda$ when I_λ is small. This is precisely what is the case for the solar spectrum; the observed energy curve lies above the black body curve of equal area near its maximum and is depressed beneath it for the smaller values on either side.

§5. *The Temperature Distribution in the Interior implied by an Observed Law of Darkening, for Gray Material.*

The agreement of the observed darkening on the sun in the separate wave-lengths with that deduced from the empirical darkening in integrated light, suggests that this question is worth pursuing further. We proceed to show how, assuming black

* The observed values show some evidence of an increase in the rate of darkening close to the limb. This is in accordance with theory. (See Paper 1, p. 372.)

body conditions, the temperature distribution can be deduced from the law of darkening in integrated light in the general case. Let ϕ $(\cos\theta)$ be the observed intensity at the point θ on the disc. Then if $B(\tau)$ is the function giving the intensity of radiation corresponding to the unknown temperature distribution, we have

$$\phi(\cos\theta) = \int_0^\infty B(\tau)\, e^{-\tau\sec\theta} \sec\theta\, d\tau,$$

or, putting $\cos\theta = \mu$,

$$\phi(\mu) = \int_0^\infty B(\mu\tau)\, e^{-\tau}\, d\tau, \quad \ldots \ldots \ldots \quad (8)$$

Hence, differentiating,

$$\phi'(\mu) = \int_0^\infty B'(\mu\tau)\, \tau e^{-\tau}\, d\tau,$$

$$\phi''(\mu) = \int_0^\infty B''(\mu\tau)\, \tau^2 e^{-\tau}\, d\tau,$$

and so on. Putting $\mu = 0$ in these in turn, we have

$$\phi(0) = \int_0^\infty B(0)\, e^{-\tau}\, d\tau = B(0),$$

$$\phi'(0) = \int_0^\infty B'(0)\, \tau e^{-\tau}\, d\tau = B'(0),$$

$$\phi^{(r)}(0) = \int_0^\infty B^{(r)}(0)\, \tau^r e^{-\tau}\, d\tau = r!\, B^{(r)}(0).$$

Hence we should expect as a formal solution of the integral equation (8),

$$\begin{aligned} B(\tau) &= B(0) + \tau B'(0) + \frac{\tau^2}{2!} B''(0) + \ldots \\ &= \phi(0) + \frac{\tau}{(1!)^2}\phi'(0) + \frac{\tau^2}{(2!)^2}\phi''(0) + \frac{\tau^3}{(3!)^2}\phi'''(0) + \ldots, \quad \ldots \quad (9) \end{aligned}$$

and it is clear that if this converges for all values of τ it is the actual solution. The temperature distribution is thus determined by the differential coefficients of the intensity at the limb.

For the observed darkening in the sun no modification of the simple cosine law gives any reasonably better approximation; to this correspond only the first two terms of (9) (as in §4), so that the application of (9) is at present limited.

We will now examine to what extent a temperature distribution given by (9) violates the condition of radiative equilibrium. For simplicity let us confine ourselves to two terms of this expansion, say,

$$\phi(\cos\theta) = B_0(1+\beta\cos\theta),$$

$$B(\tau) = B_0(1+\beta\tau).$$

Then the outward stream of radiation at any point is given by

$$\mathrm{I}(\tau,\theta) = e^{\tau\sec\theta}\int_{\tau}^{\infty}\mathrm{B}(\tau)\,e^{-\tau\sec\theta}\sec\theta\,d\tau$$

$$= \mathrm{B}_0(1+\beta\tau+\beta\cos\theta),$$

and the inward stream by

$$\mathrm{I}'(\tau,\psi) = e^{-\tau\sec\psi}\int_{0}^{\tau}\mathrm{B}(\tau)\,e^{\tau\sec\psi}\sec\psi\,d\tau$$

$$= \mathrm{B}_0\left[(1-e^{-\tau\sec\psi})(1-\beta\cos\psi)+\beta\tau\right].$$

The excess of absorption over emission at any point is proportional to

$$2\pi\int_0^{\frac{1}{2}\pi}\mathrm{I}\sin\theta\,d\theta+2\pi\int_0^{\frac{1}{2}\pi}\mathrm{I}'\sin\psi\,d\psi-4\pi\mathrm{B}.$$

Omitting the factor $2\pi\mathrm{B}_0$ and inserting for I and I′ this reduces to

$$\int_1^{\infty}\frac{\beta-\mu}{\mu^3}\,e^{-\tau\mu}\,d\mu$$

$$= e^{-\tau}\left(\tfrac{1}{2}\beta-1-\tfrac{1}{2}\beta\tau\right)+\tau\left(1+\tfrac{1}{2}\beta\tau\right)\mathrm{Ei}(\tau). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

Near the boundary (τ small) this shows that the absorption exceeds the emission or not according as $\beta \gtrless 2$. For τ large, expression (10) is given by the asymptotic formula

$$(\beta-1)\frac{e^{-\tau}}{\tau}-(3\beta-2)\frac{e^{-\tau}}{\tau^2}+\ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

and so is positive or negative according as $\beta \gtrless 1$, and vanishes for large values of τ.

These considerations enable us to say what values of β are physically permissible. The matter near the boundary may gain heat in other ways than by the absorption of radiation, *e.g.* by convection from the interior. But it can only lose heat by radiation. Hence an excess of absorption over emission, for a steady state, is impossible at the boundary, and hence $\beta \leqslant 2$. Again, the net outward flux of radiation at any point is given by

$$\pi\mathrm{F} = 2\pi\int_0^{\frac{1}{2}\pi}\mathrm{I}\sin\theta\cos\theta\,d\theta-2\pi\int_0^{\frac{1}{2}\pi}\mathrm{I}'\sin\psi\cos\psi\,d\psi$$

$$= 2\pi\,\mathrm{B}_0\left[\tfrac{2}{3}\beta+\int_1^{\infty}\frac{\mu-\beta}{\mu^4}\,e^{-\tau\mu}\,d\mu\right].$$

Hence the net radiative flux at the boundary is

$$\pi\mathrm{F}_0 = 2\pi\mathrm{B}_0\left(\tfrac{1}{3}\beta+\tfrac{1}{2}\right),$$

whilst that in the far interior tends to

$$\pi F_{\infty} = \frac{4\pi B_0 \beta}{3}.$$

In the strictly steady state, the total net flux of heat for all modes of transport must be strictly constant. This net outward flux is the sum of the radiative flux, the convective flux and the conductive flux.* Owing to the temperature gradient being positive inwards, the sum of the latter two fluxes must be positive, save at the boundary where they are zero. It follows that for a steady state to be possible

$$F_0 \geqslant F_{\infty},$$

giving

$$\beta \leqslant \tfrac{3}{2}.$$

Lastly, any convective outward flux must be provided by the excess heat accumulated by the matter in the interior; in the interior, therefore, the excess of absorption must be positive, hence by (11)

$$\beta > 1.$$

Thus, altogether, it is necessary that

$$\tfrac{3}{2} \geqslant \beta > 1; \qquad \ldots \ldots \ldots \ldots \qquad (12)$$

the coefficient of darkening u is equal to $\beta/(1 + \beta)$, whence for all physically possible states,

$$\tfrac{3}{5} \geqslant u > \tfrac{1}{2}. \qquad \ldots \ldots \ldots \ldots \qquad (13)$$

These results are only true on the somewhat specialised assumption that $B(\tau)$ is a linear function of τ, which is unlikely to be strictly true near the boundary. However, it is interesting to notice that the sun satisfies the conditions; for the sun, $u = 0 \cdot 54$ (about) and $\beta = 1 \cdot 17$. I find by numerical solution of (10) that for $\beta = 1 \cdot 17$ the excess of absorption over radiation, which is positive in the interior and negative near the boundary, vanishes for $\tau = 3 \cdot 5$. Heat must accordingly be convected from the matter inside $\tau = 3 \cdot 5$ to the matter outside it. Whether this convection is possible on other grounds, or whether there is any reason to suppose it occurs, are questions that will not be further considered here.

§6. *The Law of Darkening and Selective Absorption in the Continuous Spectrum.*

We now go on to enquire whether it is not possible to account for the observed law of darkening when the form of the spectral energy curve is taken into account.

* This type of argument was first used by GOLD, in connection with the earth's atmosphere. 'Roy. Soc. Proc.,' A, vol. 82, p. 43 (1909).

Fig. 1 shows the observed energy curves according to ABBOT* and to WILSING† respectively, for light from the disc as a whole. They have been drawn on an absolute scale of energy, so as to include an area which represents the solar constant, and the (approximately black body) curve for radiative equilibrium, enclosing the same area is shown also. (For details, see Appendix II.) Both observed curves agree in having a much more pronounced peak than the black body curve, in being depressed below

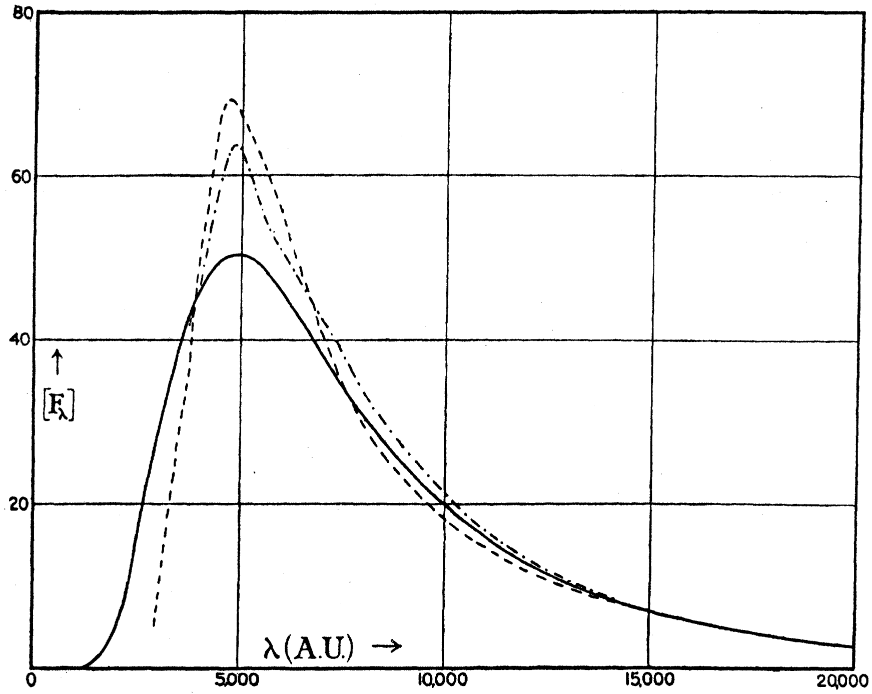

Fig. 1.—The solar energy curve. (See Appendix II.)

- - - - From observations of ABBOT, FOWLE and ALDRICH.

- · - · From observations of WILSING.

——— Theoretical curve for "gray" body in radiative equilibrium giving same net flux.

the latter in the violet (the drop in intensity on the violet side of the maximum being very sudden), and in coinciding with the black body curve in the extreme infra-red. ABBOT'S curve is also depressed beneath the black body curve on the red side of the maximum; but it must be remembered that the actual details of the relative positions of the curves depend somewhat on the allowances that have to be made for unobserved energy in the extreme infra-red and in the ultra-violet. For definiteness we shall base the argument on ABBOT'S curve.

Let us make the hypothesis that the departure from a black body curve is due to the effect of varying general absorption throughout the spectrum, *i.e.* to the departure of the properties of the material from those of a gray body. We postulate different

* 'Annals,' etc., 3, p. 197 (1913).

† "Über die Helligkeitsverteilung im Sonnenspektrum nach bolometrischen Messungen und über die Temperatur der Sonnenatmosphäre." 'Pub. Astrophys. Obs. Potsdam,' vol. 23, No. 72 (1917).

values of the mass-absorption coefficient k_λ in different parts of the spectrum; and we further assume that the material is optically homogeneous with respect to depth, so that k_λ has the same value at all depths contributing appreciable radiation (it is even immaterial if all the k_λ's change with depth proportionally to one another, but the case of separate layers at different levels absorbing differently is excluded).

In parts of the spectrum where k_λ is smaller than the average, the material is more transparent, the radiation comes from greater depths, and has, therefore, a higher temperature, and the curve rises above the black body curve. Where k_λ is greater than the average, the material is less transparent, the radiation comes only from superficial layers, and the curve is depressed below the black body curve. The relative values of k_λ from place to place in the spectrum can be deduced from the two curves. Consider now the effect on the law of darkening. Near the limb we see only into the superficial layers however large or small is the coefficient of absorption, and the intensity there should be approximately black radiation. It follows that where k_λ is small there should be a larger limb-centre contrast than predicted by the black body theory, and where k_λ is large a smaller contrast. (Indeed, where k_λ is so large as to give a dark Fraunhofer line, we should be dealing with the same temperature—the superficial temperature*—at the limb as at the centre, and the intensity in the line should show no decrease towards the limb whatever.) Consequently, we ought to find the darkening at the limb most pronounced in those wave-lengths for which the observed curve rises above the black body curve; less pronounced where it is depressed below it.

To give mathematical expression to this, write $k_\lambda/k = n_\lambda$, where k is a mean amongst the values of k_λ. Then, where there is strong absorption $n_\lambda > 1$, where the absorption is less than the average $n_\lambda < 1$. The equation of the transfer of radiation,

$$\cos\theta \frac{d\mathrm{I}_\lambda}{dx} = k_\lambda \rho\,(\mathrm{I}_\lambda - \mathrm{B}_\lambda),$$

now becomes

$$\cos\theta \frac{d\mathrm{I}_\lambda}{d\tau} = n_\lambda\,(\mathrm{I}_\lambda - \mathrm{B}_\lambda),$$

and the emergent radiation is given by

$$\mathrm{I}_\lambda(0,\theta) = \int_0^\infty \mathrm{B}_\lambda(\tau)\, e^{-n_\lambda \tau \sec\theta}\, n_\lambda \sec\theta\, d\tau$$

$$= \int_0^\infty \mathrm{B}_\lambda(\tau\cos\theta/n_\lambda)\, e^{-\tau}\, d\tau. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

In this, τ denotes the mean optical thickness up to any point, calculated in terms of k,

$$\tau = \int_0^x k\rho\, dx.$$

* *Cf.* 'M.N., R.A.S.,' 81, p. 510 (1921). (Paper 3.)

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Now let us make the further assumption that the general temperature distribution characteristic of radiative equilibrium is still preserved; this must be approximately true. In that case, provided the mean value k is properly chosen, we may regard the temperature distribution as given by

$$B = \tfrac{1}{2}F\left(1+\tfrac{3}{2}\tau\right), \qquad T^4 = T_0^{\,4}\left(1+\tfrac{3}{2}\tau\right).$$

Inserting this in (14), we have finally

$$I_\lambda(0,\,\theta) = KT_0^{\,5} f\left(\alpha,\,\tfrac{3}{2}\cos\theta/n_\lambda\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

Since $f(\alpha,\,p)$ decreases as p decreases, it is clear that $I_\lambda(0,\,\theta)$ decreases as n_λ increases, as it should.

The applicability of these ideas to the sun was tested in the following way :—

The intensity curve for the centre of the disc (as given by ABBOT) was plotted on an absolute scale against λ or $1/\alpha$, and a network of curves of $f(\alpha,\,p)$ for various values of p was drawn on the same diagram. From the position of the observed curve with respect to these the corresponding value of p for each λ could be read off, and n_λ followed from the relation $\tfrac{3}{2}/n_\lambda = p$. These values of n_λ are given in Table III. With

TABLE III.

λ.	n_λ.	λ.	n_λ.
3230	1·97	6040	0·71
3860	1·09	6700	0·76
4330	0·81	6990	0·77
4560	0·69	8660	1·16
4810	0·68	10310	1·36
5010	0·70	12250	1·02
5340	0·70		

these values the darkening-ratios were re-calculated from formula (15); they are tabulated in row (iii), Table I.

A glance at these ratios and at the observed values shows that there is no systematic improvement in the agreement. On the contrary, the agreement is definitely much worse in all cases where n_λ is appreciably different from unity; see, for example, $\lambda\lambda$ 3230, 4330, 4560, 4810, 5010, 6040, 6990. (There is perhaps some improvement for $\lambda\lambda$ 8660, 10310, so that here the depression in the energy curve may rightly be attributed to increased general absorption.) But more important than the general non-agreement is the absence of any indication that the darkening responds to the deformation of the continuous spectrum in the way suggested; there is no sign of that increased darkening in the neighbourhood of the peak which the theory anticipates.

We conclude with considerable assurance that the departure of the sun's energy spectrum from that of a black body is not due to a varying general absorption in the layers contributing the radiation. Our first attempt at correlating the darkening and the continuous spectrum has proved unsuccessful.

§7. *General Considerations on Scattering.*

We turn now to explanations based on scattering. A hopeful line of argument presents itself immediately. Suppose that outside the "effective radiating photospheric layers" (leaving this phrase somewhat undefined for the moment) there is a purely scattering atmosphere whose function is to deplete the continuous spectrum so as to give it its observed form and to reduce the finally emergent radiation to the amount given by the solar constant. Then the existence of a steady state implies, as already insisted on, that the energy robbed from the outward beam by scattering must be returned by the same scattering mechanism to the interior.

Accordingly, the scattering atmosphere will act like a blanket, keeping the radiating layers warmer than they would otherwise have been, and viewed through the scattering atmosphere these layers might be expected to show evidence of a temperature higher than that calculated from the solar constant. This is precisely what they do. If we regard the action of the scattering atmosphere as equivalent to that of a translucent screen cutting down the intensity at all points of the disc in the same ratio, we should expect that the darkening towards the limb would be the same as that of the photospheric layers themselves, corresponding to that of a black body at a higher temperature and therefore giving reduced contrast in any given wave-length; the darkening would follow the black body law for radiative equilibrium, although the observed spectrum would no longer be that of a black body, but would depend on the scattering law for wave-lengths. In any wave-length for which the scattering was small the photospheric radiation would come through in nearly its full intensity. The first of these considerations would explain the higher temperature demanded by the observed darkening, the second the peak in the energy spectrum.

A model such as that sketched would certainly give changes in the right direction, and it becomes a matter for numerical calculation. But two points must first be mentioned. The first is that the decreased contrast in any given λ will be much less than that due to increased temperature only. For the radiation structure in the photospheric layers can no longer be calculated with a boundary condition of zero incident radiation, since the back radiation from the scattering atmosphere must be taken into account; and this means that the radiation from the photosphere will be more evenly distributed in different directions. It is as though we took a section of the outward-flowing radiation some distance inside the boundary of a spherical absorbing mass in ordinary radiative equilibrium; and we know that the radiation becomes approximately isotropic at great depths.

Calculation will show that this effect is large; a comparatively thin blanketing layer effects a great reduction in the contrast. But we must not take the blanketing layer as too thin or it will be inadequate to produce the distortion of the spectrum.

The second point is the assumption that a scattering atmosphere is possible which reduces the intensity at centre and limb in the same proportion. But if it is in the form of a thin spherical shell, the path length of a photospheric ray through it will be much greater near the limb than at the centre; near the limb more light will be scattered out of the photospheric beam; and again more light will be scattered into it. The extent of the compensation is a question for calculation. It may be said at once that the degree of compensation is quite definite. A scattering layer of the kind considered tends to reduce the coefficient of darkening to about $\frac{3}{5}$ in all wave-lengths, independent of the law of darkening of the photospheric background. The effect will naturally be small for optically thin layers, but again the layer must be thick enough to be adequate in other respects. On the other hand, we could postulate an extensive scattering atmosphere of very small density whose thickness was comparable with, or larger than, the sun's radius. The amount of scattering material between the observer and the photosphere, assuming homogeneity, would then be practically the same at the limb as at the centre, and the limb and centre intensities would be cut down in the same ratio. Only, if the scattering atmosphere were too extensive, the amount of light scattered at least once would become large compared with the direct beam, and the details of the actual photospheric distribution would get smoothed out. Besides that, there would be the radiation scattered from those portions of the atmosphere lying outside the visible disc of the sun; we must make the atmosphere optically thick enough to distort the spectrum, but if we give it too much work to do we shall get too much illumination outside the visible disc.

These difficulties impair the initial attractiveness of the hypothesis of a scattering atmosphere. It is the business of the following paragraphs to analyse some of them in detail.

§8. *The Law of Darkening for a Purely Scattering Atmosphere.*

Consider for simplicity a distribution of scattering material stratified in parallel planes, through which radiation is passing. The material is supposed to be such that emission and absorption are negligible; hence the radiation traversing it must be supposed to be provided by some source which is not within the layer. Assume then that the material is bounded internally by a plane radiating and absorbing surface, and that externally it has a plane boundary on which the radiation incident from outside is zero. Assume lastly that the system is in a steady state. The radiation emergent from the external boundary will then be the radiation incident on the scattering material from the internal source less what is scattered back to the source. The state being steady, there will be across each plane of stratification a certain net outward flux of radiation, everywhere the same, and equal to the radiation emergent from the boundary. The optically thicker the scattering material, the smaller will be the net flux, for a given intensity of radiation from the internal boundary (the source); and conversely for a given net flux the source of radiation must be more intense the thicker the scattering material. We may, if we wish, consider the limiting case of an infinitely thick scattering layer, still preserving the same net flux of radiation; the radiating source must be supposed to recede to infinity, and its temperature become infinite in a certain definite way.

Though the total net flux is constant throughout the material, its distribution in direction may alter considerably from place to place. The current of outward flowing radiation depends at the internal boundary on the radiating source, and the distribution of this may be supposed arbitrarily given. As we approach the external boundary, the distribution of the radiation in direction will depend less and less on that in the source, and more and more on the laws of scattering, until for an atmosphere optically very thick the distribution of the emergent light in direction will be independent of that in the source. The distribution in direction for a plane mass being essentially the same as the law of darkening of the disc for a spherical mass, we may say that the law of darkening for a sufficiently thick scattering atmosphere depends only on the laws of scattering, and not at all on the law of darkening of the background.

The law of gaseous scattering* as found by RAYLEIGH shows that the amount of light scattered in any direction depends on that direction, and on the wave-length of the light, as well as on the properties of the medium. The amount scattered increases as the fourth power of the frequency, but the important point for our purpose is that the variation in intensity of scattered light with direction is independent of wave-length; it is proportional to $1 + \cos^2 \beta$, where β is the angle with the direction of the unscattered beam. If now the scattering atmosphere is optically thick enough to obscure the background almost completely, even for the longer wave-lengths, the law of darkening of the background is suppressed in all wave-lengths and the law of darkening of the radiation issuing from the scattering atmosphere becomes the same for all wave-lengths. The *intensity* of the emergent radiation will be much smaller in blue light than in red light, but the ratio of the intensities in any two given directions must be the same everywhere on the disc. The conclusion does not depend on whether the scattering actually follows RAYLEIGH's law; all that is necessary is that the scattering should be given by a function of wave-length multiplied by a purely trigonometrical factor.

The effects of intense scattering in modifying the law of darkening are thus to be clearly distinguished from those of absorption, selective or general. The former gives the same law for light of all colours, the latter a darkening increasing progressively from the red to the violet. We have potentially a method of distinguishing between absorption and scattering in stellar atmospheres.

This conclusion is not new; it is implicit in the paper of SCHWARZSCHILD† cited

* In this paper the discussion is confined to *gaseous* scattering. Scattering by dust particles, in so far as it differs from gaseous scattering, is not taken into account.

† "Über Diffusion und Absorption in der Sonnenatmosphäre." 'Berlin Sitz.,' 1914, p. 1183.

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in Paper 1. SCHWARZSCHILD worked out the complete distribution of the scattered light for an atmosphere such as we have been considering, on the further assumptions (1) that the background radiates equally in all directions, (2) that owing to the approximate isotropy of the radiation, save near the outer boundary, RAYLEIGH'S factor $1 + \cos^2 \beta$ may be replaced by a constant mean value, *i.e.* that the scattering particles scatter equally in all directions. The first restriction is easily removed, as we shall see, and is in any case immaterial for a thick atmosphere; the second is required to make the analysis tractable, but it is probably sufficiently near the truth. SCHWARZSCHILD obtained first an approximate solution which he termed "SCHUSTER'S approximation," since it was based on the method used by SCHUSTER in his paper* on "Radiation Through a Foggy Atmosphere" (the use of the equations of linear flow of radiation); he obtained, secondly, the correction to SCHUSTER'S approximation. The result is that for an extensive atmosphere the law of darkening is approximately a cosine law, with a coefficient of darkening of rather less than $\frac{2}{3}$ (SCHUSTER'S approximation), independent of the coefficient of scattering. The analysis is identical, in fact, with that required for radiative equilibrium in an absorbing material, as far as the integrated radiation is concerned.

The application which SCHWARZSCHILD made was to the relative intensities in a Fraunhofer line, and in the neighbouring continuous spectrum, and to the variation of the ratio of these from the centre to the limb. If a Fraunhofer line is due to selective absorption in an extensive absorbing mass, it should vanish at the limb in the continuous background; if it is due to any kind of selective scattering in a general scattering mass, then the intensities in the line and just outside the line should be cut down in the same ratio from the centre to the limb, since by the foregoing theory the darkening is independent of the magnitude of the scattering. SCHWARZSCHILD showed that the observed relative decrease in intensity in the H and K lines was more compatible with the constant ratio of line to background demanded by the scattering theory than with the limb-vanishing demanded by the absorption theory, of which there is little trace. He concluded that "it is probable that the scattering of light plays an important rôle in the phenomena of the solar atmosphere." Possibly a more detailed discussion of the processes of absorption, especially taking into account the fact that Fraunhofer lines appear to have their origins located at different levels, would modify this conclusion; it is, however, beyond the scope of the present paper to consider the question of absorption lines.

SCHWARZSCHILD confined himself to adjacent portions of the spectrum, where the wave-length was practically the same but the absorption coefficient assumed different. But there seems no reason why exactly the same reasoning should not be applied to widely separated portions of the spectrum, where the coefficients of scattering are different in virtue of the wave-lengths being different. The intensities in such portions

* 'Astrophys. Journ.,' 21, p. 1 (1905).

should be weakened, between centre and limb, in the same ratio; the law of darkening should be the same throughout the spectrum.

This is precisely what is not the case for the sun. The darkening from centre to limb is stamped with the impress of absorption and emission according to the black body law. The conclusion is irresistible that the bulk of the light emerging from the sun is not scattered light. When we are viewing the sun's surface, we are seeing on the whole the actual emitting particles, we are not looking into a "foggy atmosphere." This result is in contradiction to the conclusions of JULIUS* (which were also obtained from the evidence of darkening), and to some of the views of ABBOT and others.

§9. *The Law of Darkening for the Radiation crossing the Boundary of a Mass of Gray Material in Radiative Equilibrium into a Scattering Atmosphere.*

It is still possible that scattering may have an effect in modifying the law of darkening, even if absorption plays the predominating part. We therefore investigate the radiation structure for a purely scattering atmosphere of finite thickness in front of an absorbing and radiating mass in radiative equilibrium. The investigation divides into two parts; it is first necessary to determine the blanketing effect of the scattering layer in modifying the law of darkening of the radiation crossing from the radiating material into the scattering atmosphere; it is then necessary to determine the influence of the scattering atmosphere in modifying the law of darkening of the finally-emergent radiation. This section is devoted to the first of these.

Denote as usual by πF the constant net flux characterising the state of radiative equilibrium. Let τ be the optical thickness at any point, measured inwards into the radiating mass from the boundary between it and the scattering atmosphere; and let T'_0 be the temperature at this boundary. The boundary condition of our earlier investigations, namely, that $I'_0 = 0$, has now to be replaced by taking for I'_0 (the radiation scattered back by the scattering atmosphere) some function of direction which depends on the thickness, etc., of the scattering atmosphere. Assume I'_0 is given by an expression of the type

$$I'_0 = c-2d\cos\psi,$$

where c and d are constants and ψ is the angle with the normal. Then the equations of radiative transfer,

$$\cos\theta\frac{dI}{d\tau} = I-B, \qquad \cos\psi\frac{dI'}{d\tau} = B-I',$$

have as their solution

$$I(\tau,\theta) = e^{\tau\sec\theta}\int_\tau^\infty B(\tau)\,e^{-\tau\sec\theta}\sec\theta\,d\tau,$$

$$I'(\tau,\psi) = e^{-\tau\sec\psi}\left[\int_0^\tau B(\tau)\,e^{\tau\sec\psi}\sec\psi\,d\tau+c-2d\cos\psi\right].$$

* 'Astrophys. Journ.,' 38, p. 138 (1913).

The radiative equilibrium must resemble that in the far interior, owing to the protecting effect of the atmosphere. Assume, therefore, for the temperature function $B(\tau)$, a form

$$B(\tau) = a+2b\tau,$$

where to satisfy the condition of radiative equilibrium in the interior* we must have $b = \frac{3}{8}F$. Then

$$I(\tau,\theta) = a+2b\cos\theta+2b\tau, \quad \ldots \quad (16)$$

$$I'(\tau,\psi) = (a-2b\cos\psi)(1-e^{-\tau\sec\psi})+2b\tau+e^{-\tau\sec\psi}(c-2d\cos\psi). \quad \ldots \quad (17)$$

The condition of radiative equilibrium at the boundary demands that

$$b-d = a-c,$$

and the flux condition demands

$$\tfrac{2}{3}(b+d) = \tfrac{1}{2}F.$$

Hence $c = a$ and $d = b = \frac{3}{8}F$. The value of a (and hence of the interfacial temperature T'_0 given by $a = \sigma T'^4_0/\pi$) is undetermined, and indeed is so far quite arbitrary. It depends on the amount of radiation scattered back by the atmosphere. Let ϵ be the fraction of the radiation incident on the boundary which is returned by scattering. The incident radiation is

$$2\pi\int_0^{\frac{1}{2}\pi} I(0,\theta)\sin\theta\cos\theta\,d\theta = 2\pi(\tfrac{1}{2}a+\tfrac{2}{3}b),$$

the returned radiation is

$$2\pi\int_0^{\frac{1}{2}} I'(0,\psi)\sin\psi\cos\psi\,d\psi = 2\pi(\tfrac{1}{2}a-\tfrac{2}{3}b),$$

whence

$$\epsilon(\tfrac{1}{2}a+\tfrac{2}{3}b) = \tfrac{1}{2}a-\tfrac{2}{3}b,$$

or

$$a = \tfrac{1}{2}F\,\frac{1+\epsilon}{1-\epsilon}.$$

Since $\pi F = \sigma T_1^4$ and $T_1^4 = 2T_0^4$, we have

$$T'^4_0 = T_0^4\,\frac{1+\epsilon}{1-\epsilon}. \quad \ldots \quad (18)$$

Thus the fourth power of the interfacial temperature is $(1+\epsilon)/(1-\epsilon)$ times its value in the absence of a scattering atmosphere, for the same net flux.

Again, the radiation leaving the radiating material has been seen to be $2\pi(\frac{1}{2}a+\frac{2}{3}b)$. If we equate this to $\sigma T'^4_1$, where T'_1 is the effective temperature of the radiating material, we find

$$T'^4_1 = \frac{T_1^4}{1-\epsilon}.$$

* See Paper 1, p. 366.

The fourth power of the effective temperature of the radiating material is thus increased in the ratio $1/(1-\epsilon)$.

The law of darkening of the radiation crossing into the atmosphere is now found to be given by

$$I_\lambda(0, \theta) = KT_0'^5 \int_0^\infty \frac{\alpha'^5 e^{-\tau}\, d\tau}{e^{\alpha'(1+p'\tau)^{-\frac{1}{4}}}-1} = KT_0'^5 f(\alpha', p'), \quad . \quad . \quad . \quad . \quad (19)$$

where

$$\alpha' = \frac{hc}{\lambda R T_0'}, \qquad p' = \frac{3}{4}\frac{T_1^4}{T_0'^4}\cos\theta = \frac{3}{2}\cos\theta\,\frac{1-\epsilon}{1+\epsilon} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

If the subsequent effect of the scattering atmosphere were merely to reduce the intensity everywhere on the disc in the same proportion, limb and centre alike, equation (19) would give the law of darkening of the light finally emergent. (This is true in whatever way the reduction due to scattering varied with λ.) Notice now how the law given by (19) differs from that given by (3), the law in the absence of a scattering atmosphere. Since $T_0' > T_0$, $\alpha' < \alpha$, and so on this ground alone (19) will give a less steep intensity decrease from the centre outwards. But further, $p' < p$, which gives a decrease less steep still, and moreover the effect of this is numerically more important, since p and p' occur as coefficients of $\cos\theta$. The effect of the blanketing is not only to maintain a warmer radiating surface, but also to distribute the radiation more evenly. The first effect alters the darkening in the separate wave-lengths without altering that in the integrated radiation; the second diminishes the coefficient of darkening of the integrated radiation from $\frac{3}{5}$ to

$$\frac{2b}{a+2b} = \frac{\frac{3}{2}(1-\epsilon)}{(1+\epsilon)+\frac{3}{2}(1-\epsilon)} = \frac{3}{5}\,\frac{1-\epsilon}{1-\frac{1}{5}\epsilon}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

It is this decrease of p with increasing temperature that is ignored by LINDBLAD. Since he takes account only of the change of α he finds a much bigger consequential change of T_0 than is necessary to account for the observed darkening on the hypothesis (which he tacitly adopts) of a scattering atmosphere reducing the intensities in the same proportion all over the disc.

The selective effect of scattering in reducing the intensities in the different wave-lengths differently makes a direct application of (21) meaningless. If, indeed, we equate (21) to $0\cdot54$, the observed mean coefficient of darkening for the sun, we find $\epsilon = 0\cdot122$; this gives $T_0' = 5130°$, an increase of $6\cdot2$ per cent., and an effective temperature T_1' of the radiating material of $5940°$. The latter value is still less than those ($6130°$ and $6120°$) deduced from the observed position of λ_{max} and from the intensity at λ_{max}. But as we shall see in a moment, even this value leads to too large a diminution in the darkening.

To show the order of magnitude of these changes, row (iv) of Table I. gives the darkening for $T_0'/T_0 = 1\cdot05$, corresponding to $\epsilon = 0\cdot097$, on the assumptions stated, namely, that the scattering atmosphere reduces the intensity in a given wave-length in the same ratio at all points of the disc; these figures have been computed from formula (19). It will be seen that row (iv) gives too little darkening almost everywhere—in the first half of the table it agrees much worse than do the figures in row (i), which were calculated on the simple theory ($\epsilon = 0$); whilst in the second half, though still giving too little darkening, row (iv) gives some improvement. A fairly good fit would probably be given by $T_0'/T_0 = 1 \cdot 02$, corresponding to $T'_0 = 4930°$, $T'_1 = 5810°$, $\epsilon = 0 \cdot 040$.

The remarkable point which now emerges is, how slight a change in the solar temperature from that calculated from the solar constant is required to account for the general run of the observed darkening, when the radiation returned to the emitting layers is thus taken into account—on the assumption that the scattering atmosphere reduces the intensity everywhere in the same ratio. A high temperature such as LINDBLAD suggests, viz., 6720°, is apparently not wanted. On the other hand, an effective temperature of 5810° is incompatible with the existence of portions of the spectrum where a temperature of 6120° is indicated. Further, we still have to reconcile the fidelity with which the darkening follows the black body law with the observed non-black body spectrum.

§10. *The Law of Darkening for a Scattering Atmosphere of Finite Optical Thickness in Front of a Radiating and Absorbing Mass.*

We turn to the second investigation mentioned at the beginning of §9.

The quantity ϵ is intimately connected with the optical thickness of the scattering atmosphere; the greater the optical thickness the greater the quantity of radiation returned. The exact relationship we will determine later, but for the moment we will consider a well-known formula of SCHUSTER'S involving this same optical thickness. Denote this optical thickness measured for a certain wave-length by σ_λ, and let R_λ be the total radiation incident from the interior on the inner boundary of the scattering layer, S_λ the total emergent radiation. The formula in question* states that approximately

$$S_\lambda = \frac{R_\lambda}{1+\sigma_\lambda}. \quad \quad (22)$$

Care must be taken to avoid making an incorrect use of this formula. In the first place, it is incorrect to suppose that the obliquely emergent radiation will be given by replacing σ_λ by $\sigma_\lambda \sec \theta$. In the second place, the formula must not be held to imply that the initially incident beam is cut down in the ratio $1/(1 + \sigma_\lambda)$ during transit through the layer. The simple formula (22) somewhat obscures the fact that the emergent radiation is the sum of two parts, the depleted incident beam and the light

* In SCHUSTER'S form the denominator is written $2 + \sigma_\lambda$, but this is due to calculating σ_λ from the *mean* coefficient of scattering of a thin plane layer for all directions.

(originally derived from the incident beam) scattered into the beam; and the formula is better written in the form corresponding to its derivation,

$$S_\lambda = R_\lambda\left[\frac{1}{1+\sigma_\lambda} - e^{-2\sigma_\lambda}\right] + R_\lambda e^{-2\sigma_\lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22\text{A})$$

where the two terms denote the scattered contribution and the transmitted contribution respectively. Now this is the total radiation. When we come to discuss the distribution of this in direction, it is clear that the second term will preserve the distribution of the incident light R_λ (apart from the value of σ_λ being effectively different for different directions), whilst the first will have a distribution very nearly independent of the initial distribution and depending principally on the law of scattering; the form of (22A) makes it clear that for large values of σ_λ the incident distribution is obliterated. In particular, in order to deduce the emergent radiation for a sun supposed to be constituted of a radiating sphere surrounded by a scattering layer, it is incorrect to divide the intensity at each point of the disc due to the radiating sphere by a factor of the form $1+\sigma_\lambda$, even if this factor is made to take account of the varying thickness of the atmosphere for oblique beams.

We proceed to obtain the correct formulæ. When we are dealing with pure scattering there is no interchange of energy between the different wave-lengths and we therefore consider each one separately. But for brevity we omit the suffix λ which is to be understood as attached to the symbols s, σ, I, F, J below. Let s be the coefficient of scattering, σ the optical thickness $\int_0^x s\rho\,dx$ to any point x, $I(\sigma, \theta)$ and $I'(\sigma, \psi)$, the outward and inward intensities at any point σ, at angles θ and ψ respectively with the normal. Let σ_1 be the optical thickness of the whole layer, I_1 (a function of θ) the radiation incident on the inner boundary from the interior, πF the net flux. Assuming the radiation is scattered equally in all directions the equations are

$$\cos\theta\,\frac{dI}{\rho\,dx} = sI - \frac{s}{4\pi}\iint I\,d\omega,$$

$$\cos\psi\,\frac{dI'}{\rho\,dx} = -sI' + \frac{s}{4\pi}\iint I\,d\omega.$$

These may be written

$$\cos\theta\,\frac{dI}{d\sigma} = I - J, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$\cos\psi\,\frac{dI'}{d\sigma} = J - I', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where

$$J = \tfrac{1}{2}\left[\int_0^{\frac{1}{2}\pi} I\sin\theta\,d\theta + \int_0^{\frac{1}{2}\pi} I'\sin\psi\,d\psi\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The quantity J is proportional to the density of radiant energy at the point σ, and any radiating and absorbing matter placed at this point would take up a temperature given by $J = B = \sigma T^4/\pi$. Multiplying (23) and (24) by $\sin\theta$ and $\sin\psi$ and integrating, we have

$$\frac{d}{d\sigma}\int_0^{\frac{1}{2}\pi} I \sin\theta \cos\theta \, d\theta - \frac{d}{d\sigma}\int_0^{\frac{1}{2}\pi} I' \sin\psi \cos\psi \, d\psi = 0,$$

or

$$\int_0^{\frac{1}{2}\pi} I \sin\theta \cos\theta \, d\theta - \int_0^{\frac{1}{2}\pi} I' \sin\psi \cos\psi \, d\psi = \tfrac{1}{2}F, \quad . \quad . \quad . \quad . \quad (26)$$

since πF is the constant net flux.

First Approximation (SCHUSTER'S *Approximation*).—The method of solution is first to find J by making some simple assumption about I and I′, and then to use this in (23) and (24) to determine I and I′ more accurately. Assume that in determining J it will be sufficient* to replace I and I′ by mean values, independent of direction. Denote these by $\bar{I}$ and $\bar{I}'$. Multiplying (23) and (24) by $\sin\theta$ and $\sin\psi$, and integrating, we find

$$\tfrac{1}{2}\frac{d\bar{I}}{d\sigma} = \bar{I} - J, \qquad \tfrac{1}{2}\frac{d\bar{I}'}{d\sigma} = J - \bar{I}',$$

whilst (25) and (26) give

$$J = \tfrac{1}{2}(\bar{I} + \bar{I}'), \qquad \bar{I} - \bar{I}' = F.$$

From these we find

$$\bar{I} = F(1+\sigma), \qquad \bar{I}' = F\sigma,$$

$$J = \tfrac{1}{2}F(1+2\sigma). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

In passing, if in the expression for $\bar{I}$ we put $\sigma = \sigma_1$, we recover SCHUSTER'S formula, (22). Using now this value of J in (23) and (24), solving and using the boundary conditions $I' = 0$ for $\sigma = 0$, and $I = I_1$ for $\sigma = \sigma_1$, we find

$$I(\sigma, \theta) = \tfrac{1}{2}F(1+2\sigma+2\cos\theta) + e^{-(\sigma_1-\sigma)\sec\theta}\left[I_1 - \tfrac{1}{2}F(1+2\sigma_1+2\cos\theta)\right] \quad . \quad (28)$$

$$I'(\sigma, \psi) = \tfrac{1}{2}F(1+2\sigma-2\cos\psi) - \tfrac{1}{2}Fe^{-\sigma\sec\psi}(1-2\cos\psi). \quad . \quad . \quad . \quad . \quad . \quad (29)$$

If we put $\sigma = 0$ in (28) we have the distribution of the escaping radiation, except that we still have to determine F in terms of I_1 supposed given.

An approximate evaluation of F we have already had, namely, $\bar{I}_1 = F(1+\sigma_1)$. To obtain a better one put $\sigma = \sigma_1$ in (28) and (29), and insert the values of I and I′ in the net flux condition at the inner boundary, namely, in the relation

$$2\pi\int_0^{\frac{1}{2}\pi} I(\sigma_1, \theta)\sin\theta\cos\theta\,d\theta - 2\pi\int_0^{\frac{1}{2}\pi} I'(\sigma_1, \psi)\sin\psi\cos\psi\,d\psi = \pi F.$$

* We follow here SCHWARZSCHILD'S procedure. SCHWARZSCHILD, however, takes I_1 independent of θ and confines himself, in essence, to large values of σ_1.

We then find on simplification (setting $\sec\psi = u$ in one of the integrals),

$$\mathrm{F}\left[\tfrac{5}{6}+\sigma_1-\int_1^\infty \frac{\mu-2}{\mu^4}e^{-\sigma_1\mu}\,d\mu\right] = 2\int_0^{\frac{1}{2}\pi} \mathrm{I}_1 \sin\theta\cos\theta\,d\theta. \quad (30)$$

We shall now discuss the evaluation of F by means of (30) for the cases σ_1 large and σ_1 small, and we shall then employ these evaluations in (28) to obtain the law of darkening.

(*a*) When σ_1 is large, (30) approximates to

$$\mathrm{F} = \frac{2}{\frac{5}{6}+\sigma_1}\int_0^{\frac{1}{2}\pi} \mathrm{I}_1 \sin\theta\cos\theta\,d\theta, \quad (31)$$

or

$$\text{net flux} = \frac{\text{incident radiation}}{\frac{5}{6}+\sigma_1}.$$

But at the outer boundary, equation (28) with $\sigma = 0$ gives for the next flux

$$2\pi\int_0^{\frac{1}{2}\pi} \mathrm{I}\,(0,\ \theta)\sin\theta\cos\theta\,d\theta = \tfrac{7}{6}\pi\mathrm{F},$$

approximately, when σ_1 is large, instead of πF. This discrepancy is an imperfection of this particular approximation.*

(*b*) When σ_1 is small, we have

$$\int_1^\infty \frac{\mu-2}{\mu^4}e^{-\sigma_1\mu}\,d\mu = \int_1^\infty \frac{\mu-2}{\mu^4}(1-\sigma_1\mu)\,d\mu + O\,(\sigma_1^2)$$
$$= -\tfrac{1}{6}+O\,(\sigma_1^2),$$

and hence, up to and including the first power of σ_1, (30) approximates to

$$\mathrm{F} = \frac{2}{1+\sigma_1}\int_0^{\frac{1}{2}\pi} \mathrm{I}_1 \sin\theta\cos\theta\,d\theta, \quad (32)$$

or

$$\text{net flux} = \frac{\text{incident radiation}}{1+\sigma_1}.$$

Using this value of F in (28) with $\sigma = 0$ we find for the flux at the outer boundary

$$2\pi\int_0^{\frac{1}{2}\pi} \mathrm{I}\,(0,\ \theta)\sin\theta\cos\theta\,d\theta = \pi\mathrm{F}\left[1+2\sigma_1\left(1-\frac{1}{\mathrm{F}}\int_0^{\frac{1}{2}\pi} \mathrm{I}_1\sin\theta\,d\theta\right)\right], \quad (33)$$

as far as the first power of σ_1. This is sufficiently near the correct value πF, σ_1, being small. (Here and in (*a*) above we are testing the equality of the net flux at the inner and outer boundaries as a check on the accuracy of the approximations.)

* *Cf.* Paper 1, p. 364. The Schuster approximation for scattering is exactly analogous to the Schwarzschild approximation for radiative equilibrium.

Resulting darkening.—(*a*) When σ_1 is large, $e^{-\sigma_1}$ can be neglected and (28) gives for the emergent radiation

$$I(0, \theta) = \tfrac{1}{2}F(1+2\cos\theta). \qquad (34)$$

This gives a coefficient of darkening equal to $\frac{2}{3}$, independent of the incident radiation.

(*b*) When σ_1 is small, the law of darkening depends partly on the distribution of the incident radiation I_1. Assume this to be such that if the scattering atmosphere were removed the surface would have a coefficient of darkening u_1. Then

$$I_1 = c(1-u_1+u_1\cos\theta), \qquad (35)$$

where c is some constant. Inserting this value of I_1 in (32) we find

$$c = F(1+\sigma_1)/(1-\tfrac{1}{3}u_1),$$

Inserting now for I_1 in (28) in terms of F and putting $\sigma = 0$, we find for the emergent radiation

$$I(0, \theta) = \tfrac{1}{2}F(1+2\cos\theta)+\tfrac{1}{2}Fe^{-\sigma_1\sec\theta}\left[\frac{2(1+\sigma_1)}{1-\frac{1}{3}u_1}(1-u_1+u_1\cos\theta)-(1+2\sigma_1+2\cos\theta)\right] \qquad (36)$$

This can also be written in the form

$$I(0, \theta) = F\frac{1+\sigma_1}{1-\frac{1}{3}u_1}(1-u_1+u_1\cos\theta)e^{-\sigma_1\sec\theta}+\tfrac{1}{2}F[(1+2\cos\theta)(1-e^{-\sigma_1\sec\theta})-2\sigma_1 e^{-\sigma_1\sec\theta}] \qquad (36\text{A})$$

Since σ_1 is small, the second term in (36A) is small compared with the first, except when θ is near $\frac{1}{2}\pi$. Accordingly, except near the limb the law of darkening is practically that of the incident radiation, with coefficient of darkening u_1. Comparing the intensities at the centre and the limb, we find approximately

$$\frac{I(0, \frac{1}{2}\pi)}{I(0, 0)} = \tfrac{1}{2}(1-\tfrac{1}{3}u_1)[1-\tfrac{1}{2}\sigma_1(1-\tfrac{1}{3}u_1)], \qquad (37)$$

so that the coefficient of darkening u to which the limb-centre ratio corresponds is given by (see footnote, §4)

$$u = 1-\frac{I(0, \frac{1}{2}\pi)}{I(0, 0)} = \tfrac{1}{2}+\tfrac{1}{6}u_1+\tfrac{1}{4}\sigma_1(1-\tfrac{1}{3}u_1)^2.$$

Writing this in the form

$$u-\tfrac{3}{5} = \tfrac{1}{6}(u_1-\tfrac{3}{5})+\tfrac{1}{4}\sigma_1(1-\tfrac{1}{3}u_1)^2, \qquad (38)$$

and remembering that σ_1 is small, we see that near the limb the effect of the thin scattering atmosphere is to alter the equivalent coefficient of darkening in such a way as to make it approach the value $\frac{3}{5}$.

*Second Approximation.**—To obtain a more accurate solution we use cosine approximations for I and I′ in determining J, instead of mean values. Assume for this purpose

$$\mathrm{I} = a+2b\cos\theta, \qquad \mathrm{I}' = a-2b\cos\psi.$$

We then find (omitting the details),

$$b = \tfrac{3}{8}\mathrm{F}, \qquad \mathrm{J} = a = \tfrac{1}{2}\mathrm{F}\,(1+\tfrac{3}{2}\sigma), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27')$$

using as the boundary condition the fact that the *total* incident radiation is zero. Hence

$$\mathrm{I}\,(\sigma,\theta) = \tfrac{1}{2}\mathrm{F}\,(1+\tfrac{3}{2}\sigma+\tfrac{3}{2}\cos\theta)+e^{-(\sigma_1-\sigma)\sec\theta}\,[\mathrm{I}_1-\tfrac{1}{2}\mathrm{F}\,(1+\tfrac{3}{2}\sigma_1+\tfrac{3}{2}\cos\theta)], \quad . \quad . \quad (28')$$

$$\mathrm{I}'\,(\sigma,\psi) = \tfrac{1}{2}\mathrm{F}\,(1+\tfrac{3}{2}\sigma-\tfrac{3}{2}\cos\psi)-\tfrac{1}{2}\mathrm{F}e^{-\sigma\sec\psi}\,[1-\tfrac{3}{2}\cos\psi). \quad . \quad . \quad . \quad . \quad . \quad (29')$$

The net flux condition at the inner boundary gives

$$\mathrm{F}\left[1+\tfrac{3}{4}\sigma_1-\int_1^\infty\left(1-\frac{3}{2\mu}\right)e^{-\sigma_1\mu}\frac{d\mu}{\mu^3}\right] = 2\int_0^{\frac{1}{2}\pi}\mathrm{I}_1\sin\theta\cos\theta\,d\theta. \quad . \quad . \quad . \quad (30')$$

(*a*) When σ_1 is large, this approximates to

$$\mathrm{F} = \frac{2}{1+\frac{3}{4}\sigma_1}\int_0^{\frac{1}{2}\pi}\mathrm{I}_1\sin\theta\cos\theta\,d\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31')$$

or

$$\text{net flux} = \frac{\text{incident radiation}}{1+\frac{3}{4}\sigma_1}.$$

The advantage of the present approximation is that (28′) will now be found to give the correct net flux, πF, at the outer boundary.

(*b*) When σ_1 is small, we have

$$\int_1^\infty\left(1-\frac{3}{2\mu}\right)e^{-\sigma_1\mu}\frac{d\mu}{\mu^3} = -\tfrac{1}{4}\sigma_1+O\,(\sigma_1^{\,2}),$$

and hence, up to and including the first power of σ_1 (30′) approximates to

$$\mathrm{F} = \frac{2}{1+\sigma_1}\int_0^{\frac{1}{2}\pi}\mathrm{I}_1\sin\theta\cos\theta\,d\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32')$$

exactly as for the first approximation. Similarly (33) holds for the total emergent radiation.

Resulting Darkening.—(*a*) When σ_1 is large (28′) gives for the emergent radiation

$$\mathrm{I}\,(0,\theta) = \tfrac{1}{2}\mathrm{F}\,(1+\tfrac{3}{2}\cos\theta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34')$$

This gives a coefficient of darkening equal to $\frac{3}{5}$, independent as before of the incident radiation.

* The analogue of (2) above for radiative equilibrium. *Cf.* Paper 1, p. 369.

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(*b*) When σ_1 is small, the law of darkening depends as before on the incident radiation I_1. If this has a coefficient of darkening u_1 we find, using (32′), that (35) still holds, and so

$$I(0, \theta) = \tfrac{1}{2}F(1+\tfrac{3}{2}\cos\theta)+\tfrac{1}{2}Fe^{-\sigma_1\sec\theta}\left[\frac{2(1+\sigma_1)}{1+\frac{1}{3}u_1}(1-u_1+u_1\cos\theta)-(1+\tfrac{3}{2}\sigma_1+\tfrac{3}{2}\cos\theta)\right], \quad (36')$$

$$\frac{I(0, \frac{1}{2}\pi)}{I(0, \theta)} = \tfrac{1}{2}(1-\tfrac{1}{3}u_1)[1-\sigma_1(1+\tfrac{7}{6}u)]. \quad (37')$$

The discussion of (37′) is essentially the same as that of (37).

Thus our second approximation leads to the same results as those already obtained from the first approximations. The only significant change made by the second approximation is the correction of the coefficient of darkening for a thick scattering atmosphere from $\frac{2}{3}$ to $\frac{3}{5}$.

It may be noticed that equation (30′) gives the relation between ϵ and σ_1. For

$$\pi F = 2\pi(1-\epsilon)\int_0^{\frac{1}{2}\pi} I_1 \sin\theta\cos\theta\, d\theta,$$

and hence

$$(1-\epsilon)\left[1+\tfrac{3}{4}\sigma_1-\int_1^{\infty}\left(1-\frac{3}{2\mu}\right)e^{-\sigma_1\mu}\frac{d\mu}{\mu^3}\right] = 1. \quad (38)$$

When σ_1 is large, $\epsilon = \sigma_1/(\frac{4}{5}+\sigma_1)$ when σ_1 is small $\epsilon = \sigma_1/(1+\sigma_1)$. Here, however, σ_1 is a function of wave-length and hence so is ϵ; the "ϵ" of §9 refers to the integrated radiation.

Numerical Results.—The radiation I_1 is that supplied by the radiating mass. It varies with θ in a separate way for each separate wave-length. Now the darkening in each separate wave-length for radiative equilibrium is not expressed so accurately by a cosine law as is the integrated radiation, and consequently there is strictly speaking no u_1. However, we can take a mean value of u_1 for each λ without serious error. It then becomes a matter of interest to determine numerically from the formulæ the modification produced in the law of darkening by a scattering atmosphere, for given u_1 and σ_1, *i.e.* for given colour and given thickness.

Table IV gives the results for a number of cases for $\sigma_1 = 0\cdot5$ and $\sigma_1 = 1\cdot0$, which may be taken to be intermediate between theoretically large and theoretically small values. The ratios are calculated from formula (28′) with $\sigma = 0$, the relation between F and I_1 being evaluated from equation (30′) by means of Ei functions. The rows named "incident" and "emergent" give the darkening in the incident and emergent radiations respectively. The rows marked I_0/I_1 show the proportional reduction in intensity effected by the scattering atmosphere at each point of the disc. The last two columns are put in for the sake of completeness, but they correspond to places on the disc too near the limb to be the subject of experiment.

TABLE IV.—Darkening Due to a Scattering Atmosphere.

σ_1	u_1	$\sin\theta$ / $\cos\theta$	0·000 / 1·000	0·600 / 0·800	0·800 / 0·600	0·866 / 0·500	0·917 / 0·400	0·954 / 0·300	0·980 / 0·200	0·995 / 0·100	1·000 / 0·000
0·5	0·4	Incident	1·000	0·920	0·840	0·800	0·760	0·720	0·680	0·640	0·600
		Emergent	1·000	0·889	0·775	0·716	0·665	0·593	0·527	0·463	0·402
		I_0/I_1	0·770	0·744	0·710	0·689	0·665	0·637	0·598	0·556	0·515
0·5	0·6	Incident	1·000	0·880	0·760	0·700	0·640	0·580	0·520	0·460	0·400
		Emergent	1·000	0·855	0·717	0·652	0·591	0·534	0·483	0·433	0·377
		I_0/I_1	0·758	0·736	0·715	0·707	0·700	0·698	0·704	0·713	0·715
0·5	0·7	Incident	1·000	0·860	0·720	0·650	0·580	0·510	0·440	0·370	0·300
		Emergent	1·000	0·840	0·690	0·621	0·559	0·505	0·461	0·418	0·364
		I_0/I_1	0·752	0·733	0·720	0·717	0·723	0·744	0·786	0·847	0·912
0·5	0·8	Incident	1·000	0·840	0·680	0·600	0·520	0·440	0·360	0·280	0·200
		Emergent	1·000	0·821	0·660	0·590	0·527	0·475	0·437	0·402	0·352
		I_0/I_1	0·745	0·729	0·723	0·732	0·754	0·805	0·906	1·070	1·310
0·5	1·0	Incident	1·000	0·800	0·600	0·500	0·400	0·300	0·200	0·100	0·000
		Emergent	1·000	0·786	0·599	0·521	0·457	0·420	0·390	0·371	0·325
		I_0/I_1	0·733	0·720	0·732	0·763	0·838	1·024	1·464	2·72	∞
1·0	0·6	Incident	1·000	0·880	0·760	0·700	0·640	0·580	0·520	0·460	0·400
		Emergent	1·000	0·848	0·713	0·650	0·594	0·540	0·488	0·433	0·376
		I_0/I_1	0·600	0·580	0·563	0·558	0·557	0·560	0·565	0·565	0·565
1·0	0·8	Incident	1·000	0·840	0·680	0·600	0·520	0·440	0·360	0·280	0·200
		Emergent	1·000	0·821	0·670	0·607	0·553	0·507	0·462	0·410	0·357
		I_0/I_1	0·581	0·568	0·572	0·588	0·618	0·670	0·745	0·851	1·036

The table shows, first, that the distribution of the radiation is little changed if the incident radiation has a darkening coefficient $u_1 = \frac{3}{5}$; the darkening is slightly enhanced. But as soon as u_1 departs from this value, the two distributions begin to differ, the sense of the difference being in the direction of bringing the emergent radiation into uniformity with $u = \frac{3}{5}$. Careful scrutiny of this table in conjunction with Table I. fails to reveal any tendency of the solar darkening in this direction. To fix the ideas, the darkening in $\lambda 6000$ in the sun corresponds roughly to $u = 0{\cdot}6$, $\lambda 4350$ to $0{\cdot}8$ and $\lambda 9500$ to $0{\cdot}4$. Scattering should cause $\lambda 4350$ to be less darkened than if the scattering layer were absent, $\lambda 9500$ to be more darkened. After taking due account of row (iv), Table I., the tendency is, if anything, in the opposite direction.

The values of I_0/I_1 given in Table IV. are of considerable interest, for they are of the precise order of magnitude of the mean reduction in intensity due to scattering hypothecated by various writers; this shows that the values of σ_1 taken are appropriate ones. LINDBLAD, for example, reduces his spectral curve calculated for 6720° by the factor 0·74, in order to get coincidence with the observed curve at $\lambda 6000$; the same applies to the suggestion of ABBOT and others that the steepness of the red side of the energy curve implies a temperature of the order of 7000°, for, drawn on an absolute scale, the curve for 7000° towers formidably above the observed curve. Now the fact of the observed darkening following the computed law in the neighbourhood of $\lambda 6000$ is compatible with the scattering theory; but in the violet, where the depression* is very much greater than is given by a factor $0 \cdot 74$, much greater values of σ_1 would be required, and discrepancies incomparably larger than the existing ones should occur, in the direction of reducing the darkening towards $u = 0 \cdot 6$.

The conclusion is that the darkening on the solar disc shows no evidence of the existence of a purely scattering atmosphere above the radiating material.

Note.—It is difficult to make direct estimates of the coefficients of scattering in the solar atmosphere, but as far as the numerical results go they confirm the negligible importance of scattering (see *e.g.* LINDBLAD, *loc. cit.*, p. 21); the densities are too small.† But this is not to deny the probability of scattering playing an important part in other stars. Indeed, there is considerable evidence to the contrary. The star γ Cassiopeiæ‡ has a spectrum of type B 0*p*, with the Balmer lines $H\alpha$ to $H\zeta$ bright (and, except $H\alpha$, centrally reversed), and with a number of bright metallic lines, thus a spectrum with distinctly chromospheric features. The suggestion is that the star is in fact provided with a very extensive chromosphere. SCHUSTER§ showed that when there is a continuous background bright lines can only be produced if there is sufficient scattering to dim the background, combined with a low temperature gradient, otherwise the lines should be absorption lines. The inference is that γ Cassiopeiæ possesses an extensive hydrogen atmosphere in which the general absorption is small compared with the scattering and the selective emission. Now the temperature of this star has been determined by two different observers using different experimental methods. H. H. PLASKETT,‖ using the neutral wedge method, determined the temperature from the position of maximum as 6800°, and drew attention to the lowness of this compared with the accepted mean for B-type stars. But the Potsdam observers¶ had just previously also determined the apparent temperature, from the form of the energy curve, and found the same value 6800°; the mean temperature of the B stars in the same list is about 10,700°. Again, MERRILL,** commenting on COBLENTZ's measures of the total radiation from the stars†† remarks: "It is interesting to notice that the bright

* If the depression is due to an accumulation of Fraunhofer lines, we have the case of §6, and the darkening should be still further reduced.

† *Cf.* GOUY, "Mesures spectrophotométriques et applications à la physique solaire," 'Ann. de Phys.,' 13, p. 215 (1920).

‡ See CURTISS's summary, 'Detroit Obs. Pub.,' 2, p. 31 (1916).

§ *Loc. cit.*

‖ 'M.N., R.A.S.,' 80, p. 771 (1920).

¶ WILSING, "Effective Temperaturen von 199 helleren Sternen." 'Potsdam Pub.,' vol. 23, No. 74 (1919).

** 'P.A.S.P.,' 27, p. 122 (1915).

†† 'L.O.B.,' 8, No. 266, p. 104 (1915), or 'Bull. Bur. Stand.,' 11, p. 613 (1914–15).

line star γ Cassiopeiæ gives a low value of the emissivity compared with other stars of class B." Here "emissivity" denotes the ratio of total radiation to luminous radiation, which for black body emission has a minimum near 7000°. Thus the apparently low temperature is amply confirmed. PLASKETT, among other explanations of this anomaly, suggested that "the true maximum might be obscured by scattering in the star's atmosphere." But there would appear to be little need to look for further explanations. When we correlate the existence of bright lines with the existence of an apparently low temperature the anomaly disappears. If molecular scattering is adequate to permit bright lines, it should be adequate to cause distortion of the continuous spectrum by the relatively increased scattering in the blue, and the result of this should be to displace the maximum to the red, giving a spectral curve not unlike a black body curve of a lower temperature. (In attempting to account for the solar spectrum by calculating the effect of scattering according to RAYLEIGH'S law, I have been much struck with the magnitude of this displacement, for quite reasonable amounts of scattering.) If now this explanation is accepted, we can predict that if ever the law of darkening can be determined for γ Cassiopeiæ it should be nearly the same in all wave-lengths.

§11. *Mixture of Scattering and Absorption: the Effect on the Continuous Spectrum.*

Thus far we have considered a purely scattering atmosphere in front of a purely radiating mass. For completeness it is necessary to examine the case in which scattering and absorption are taking place together. The equations now become complicated, and it appears that, as a preliminary to solving them, the general effect of scattering in modifying the continuous spectrum must be determined. This we accordingly do.

A state of radiative equilibrium is still assumed to hold, and the temperature distribution depends, in much the same way as before, on the optical thickness measured up to any point. But the optical thickness is now calculated from the sum of the absorption and scattering coefficients. For a given net flux the temperature at a given optical thickness below the surface is the same as in the absence of scattering, but the density of *radiating* matter around this point being relatively smaller, the radiation near the point has on the whole been emitted from more interior points. If the scattering were independent of λ, the radiation would consequently be bluer. Put another way, if the optical thickness is reckoned from the radiating matter only, the increased resistance offered by scattering to the passage of the net flux demands higher gradients, and, therefore, radiation of a higher temperature will emerge from the surface; the total amount of radiation will be unaltered, but it will contain a bigger proportion of the shorter wave-lengths. The effect will be masked if the scattering follows RAYLEIGH'S law, but it appears quite clearly in the analysis.

Adopting the approximate form of the equations of radiative transfer (equations of linear flow), we have

$$\frac{1}{2}\frac{dI_\lambda}{\rho dx} = (k_\lambda + s_\lambda) I_\lambda - k_\lambda B_\lambda - \tfrac{1}{2} s_\lambda (I_\lambda + I'_\lambda) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

$$\frac{1}{2}\frac{dI'_\lambda}{\rho dx} = -(k_\lambda + s_\lambda) I_\lambda + k_\lambda B_\lambda + \tfrac{1}{2} s_\lambda (I_\lambda + I'_\lambda). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

whilst the equation of radiative equilibrium gives

$$\int_0^\infty (I_\lambda + I'_\lambda - 2B_\lambda)\, d\lambda = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

We will first solve these for a simple case. Suppose that k_λ and s_λ are both independent of λ, and that s_λ is small compared with k_λ. Omitting the suffixes, put $s/(s+k) = \eta$, and write

$$\tau = \int_0^x (k+s)\, \rho dx, \qquad t = 2\tau.$$

Then (39) and (40) become

$$\frac{dI_\lambda}{dt} = I_\lambda - (1-\eta) B_\lambda - \tfrac{1}{2}\eta (I_\lambda + I'_\gamma). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

$$\frac{dI'_\lambda}{dt} = -I'_\lambda + (1-\eta) B_\lambda + \tfrac{1}{2}\eta (I_\lambda + I'_\lambda). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Now if (42) and (43) are integrated with respect to λ, the terms in η cut out in virtue of (41) and the equations become identical with those in the absence of scattering. Hence the temperature for any given t is the same as for pure radiative equilibrium without scattering, for a given net flux. This suggests that we should try to solve (42) and (43) by the method of small variations, by comparison with the case of no scattering. The equations for the latter case are

$$\frac{dI_\lambda}{dt} = I_\lambda - B_\lambda, \qquad \frac{dI'_\lambda}{dt} = B_\lambda - I'_\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad (44), (45)$$

where B_λ is the same function of t as in (42) and (43).

Replacing I_λ, I'_λ by $I_\lambda + \delta I_\lambda$, $I'_\lambda + \delta I'_\lambda$ in (42) and (43) and subtracting (44) and (45) we find

$$\frac{d\,(\delta I_\lambda)}{dt} = \delta I_\lambda - \tfrac{1}{2}\eta (I_\lambda + I'_\lambda - 2B_\lambda), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

$$\frac{d\,(\delta I'_\lambda)}{dt} = -\delta I'_\lambda + \tfrac{1}{2}\eta (I_\lambda + I'_\lambda - 2B_\lambda)\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

whence, solving,

$$\delta I_\lambda = \tfrac{1}{2}\eta e^t \int_t^\infty (I_\lambda + I'_\lambda - 2B_\lambda)\, e^{-t}\, dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Since η is small we may obtain a first-order solution by using in (48) the values of I_λ and I'_λ derived from (44), (45), namely,

$$I_\lambda = e^t \int_t^\infty Be^{-t} dt, \qquad I'_\lambda = e^{-t}\int_0^t Be^{+t}\,dt.$$

Substituting in (48) and removing the repeated integrals by integrations by parts, we find on putting $t = 0$,

$$(\delta I_\lambda)_0 = \tfrac{1}{2}\eta\int_0^\infty (t-\tfrac{3}{2})\, B_\lambda e^{-t}\, dt$$

$$= \tfrac{1}{2}\eta K T_0{}^5\alpha^5\int_0^\infty \frac{(t-\frac{3}{2})\, e^{-t}\, dt}{e^{\alpha(1+t)^{-\frac{1}{4}}}-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

This gives the disturbance in the emergent radiation due to scattering. It may be verified that $\delta I = \int \delta I_\lambda d\lambda$ is zero. Now it can be proved that when α is sufficiently small (λ large) the integral in (49) is negative, and that when α is sufficiently large (λ small) it is positive. This means that the effect of scattering is to increase the blue at the expense of the red; the peak is shifted slightly towards the blue. The result is only true provided the scattering is the same for all λ, but it is given to illustrate the remarks above.

We proceed to solve the general equations (39), (40), (41). Let us assume that the temperature distribution compatible with radiative equilibrium is of the form

$$B = \tfrac{1}{2}F_0(1+2\tau) = \tfrac{1}{2}F_0(1+t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

where

$$\tau = \int_0^x (\bar{k}+\bar{s})\,\rho dx, \qquad t = 2\tau;$$

here $\bar{k}$ and $\bar{s}$ are suitable means amongst the values of k_λ and s_λ, selected so that (50) is as good an approximation as possible. Write now

$$\frac{k_\lambda+s_\lambda}{\bar{k}+\bar{s}} = n_\lambda, \qquad \frac{k_\lambda}{k_\lambda+s_\lambda} = \xi_\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

and assume that n_λ and ξ_λ can be treated as functions of λ independent of t. Equations (39) and (40) become

$$\frac{dI_\lambda}{dt} = n_\lambda(I_\lambda - B_\lambda) - \tfrac{1}{2}(1-\xi_\lambda)\, n_\lambda(I_\lambda + I'_\lambda - 2B_\lambda) \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$\frac{dI'_\lambda}{dt} = -n_\lambda(I'_\lambda - B_\lambda) + \tfrac{1}{2}(1-\xi_\lambda)\, n_\lambda(I_\lambda + I'_\lambda - 2B_\lambda) \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Adding and subtracting,

$$\frac{d(I_\lambda + I'_\lambda)}{dt} = n_\lambda(I_\lambda - I'_\lambda) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

$$\frac{d(I_\lambda - I'_\lambda)}{dt} = \xi_\lambda n_\lambda(I_\lambda + I'_\lambda - 2B_\lambda). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

From these,

$$\frac{d^2(\mathrm{I}_\lambda+\mathrm{I}'_\lambda)}{dt^2} = \xi_\lambda n^2_\lambda(\mathrm{I}_\lambda+\mathrm{I}'_\lambda-2\mathrm{B}_\lambda). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Since B is known as a function of t, so is B_λ, and (46) may be solved as a second-order equation in $\mathrm{I}_\lambda+\mathrm{I}'_\lambda$. Writing down its solution, and then using (55) to find $\mathrm{I}_\lambda-\mathrm{I}'_\lambda$, we have, on setting $\xi_\lambda^{\frac{1}{2}}=\omega_\lambda$

$$\mathrm{I}_\lambda+\mathrm{I}'_\lambda = \mathrm{P}e^{-n_\lambda\omega_\lambda t}+\mathrm{Q}e^{n_\lambda\omega_\lambda t}+n_\lambda\omega_\lambda\left[e^{n_\lambda\omega_\lambda t}\int_t^\infty \mathrm{B}_\lambda e^{-n_\lambda\omega_\lambda t}+e^{-n_\lambda\omega_\lambda t}\int_0^t \mathrm{B}_\lambda e^{n_\lambda\omega_\lambda t}\right], \quad . \quad . \quad (57)$$

$$\mathrm{I}_\lambda-\mathrm{I}'_\lambda = \omega_\lambda(-\mathrm{P}e^{-n_\lambda\omega_\lambda t}+\mathrm{Q}e^{n_\lambda\omega_\lambda t})+n_\lambda\omega_\lambda\left[e^{n_\lambda\omega_\lambda t}\int_t^\infty \mathrm{B}_\lambda e^{-n_\lambda\omega_\lambda t}-e^{-n_\lambda\omega_\lambda t}\int_0^t \mathrm{B}_\lambda e^{n_\lambda\omega_\lambda t}\right]. \quad (58)$$

Here P, Q are constants to be found from the boundary conditions, namely, that when $t=0$, $\mathrm{I}'_\lambda=0$, and that when $t\to\infty$, I_λ (or I'_λ) is at most of the order of magnitude of B_λ. These conditions give

$$\mathrm{Q}=0, \qquad \mathrm{P}=-\frac{1-\omega_\lambda}{1+\omega_\lambda}\int_0^\infty \mathrm{B}_\lambda e^{-n_\lambda\omega_\lambda t}n_\lambda\omega_\lambda dt. \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Inserting these in (57) and (58), solving for I_λ and putting $t=0$, we have finally for the spectral distribution of the emergent light

$$\begin{aligned}\mathrm{I}_\lambda(0) &= \frac{2\omega_\lambda}{1+\omega_\lambda}\int_0^\infty \mathrm{B}_\lambda(t)\, e^{-n_\lambda\omega_\lambda t}\, n_\lambda\omega_\lambda\, dt \\ &= \frac{2\omega_\lambda}{1+\omega_\lambda}\int_0^\infty \mathrm{B}_\lambda(t/\omega_\lambda n_\lambda)\, e^{-t}\, dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)\end{aligned}$$

From this simple formula it is possible readily to deduce the various results which SCHUSTER* obtained in his paper, "Radiation Through a Foggy Atmosphere," concerning the relative brightness and darkness of different parts of the spectrum, for different relative values of k_λ, s_λ, $\bar{k}$ and $\bar{s}$. SCHUSTER, however, took no account of the radiation lost by absorption, whilst (60) with the temperature distribution given by (50) pays due attention to the radiative equilibrium.

The equation we ultimately make use of is (57). We shall, however, pause to make a curious deduction from (60) which is of some mathematical interest. Assume that there is no selective absorption or selective scattering, *i.e.* $n_\lambda=1$, $\omega_\lambda=$ const. Then the total emergent radiation is given by

$$\mathrm{I}(0)=\frac{2\omega}{1+\omega}\int_0^\infty \mathrm{B}(t/\omega)\,e^{-t}dt=\frac{2\omega}{1+\omega}\int_0^\infty \tfrac{1}{2}\mathrm{F}\left(1+\frac{t}{\omega}\right)e^{-t}dt=\mathrm{F},$$

as it should be; the point to notice is that it is constant and independent of ω. For any given λ, however, $\mathrm{I}_\lambda(0)\to 0$ as $\omega\to 0$; it is found in fact from (60) to be of the

* *Loc. cit.*

order of magnitude of $\omega^{\frac{3}{4}}$. Now as $\omega \rightarrow 0$, $s/k \rightarrow \infty$. Consequently the result implies that as the scattering increases relatively to the absorption, any given small portion of the spectral energy curve tends to approach coincidence with the axis of λ. But the area included between the whole curve and the axis of λ remains constant, being equal to F. What becomes, then, of the characteristic form of the energy curve ? We have seen that for *small* amounts of scattering the peak tends to move towards the blue (equation (49)). We are now going to prove that, as the scattering increases, the peak moves steadily towards the blue and the intensity at the peak tends to infinity, *i.e.* that $I_\lambda(0)$ tends to zero *non-uniformly* as ω tends to zero.

Consider the asymptotic formula*

$$f(\alpha, p) \sim e^{1/p} \alpha^5 \left(\frac{8\pi}{5p}\right)^{\frac{1}{2}} \left(\tfrac{1}{4}\alpha p\right)^{\frac{2}{5}} e^{-5p^{-1}\left(\frac{1}{4}\alpha p\right)^{\frac{4}{5}}} . \qquad (61)$$

Let us use this tentatively to determine the value of $\alpha_{\max}$. Differentiating for a maximum we find

$$\alpha_{\max} = \left(\tfrac{27}{5}\right)^{\frac{5}{4}} \left(\tfrac{1}{4}p\right)^{\frac{1}{4}} = 5 \cdot 81 p^{\frac{1}{4}}. \qquad (62)$$

Now (61) is valid only if $\alpha p^{-\frac{1}{4}}$ is sufficiently large. But from Paper 2, p. 381, when $p = 1$, $\alpha_{\max} = 6 \cdot 09$, so that (62) gives a reasonable approximation in this case. We may therefore assume that $\alpha p^{-\frac{1}{4}} = 6$ is a value sufficiently large for (61) to be a useful approximation, and hence that (62) gives a reasonable approximation for $\alpha_{\max}$ in all cases. The value of $f(\alpha, p)$ at $\alpha = \alpha_{\max}$ is accordingly

$$e^{1/p} \left(\frac{8\pi}{5p} \cdot \frac{27}{5} \cdot \frac{p}{4}\right)^{\frac{1}{2}} \left(\tfrac{27}{5}\right)^{\frac{29}{4}} \left(\tfrac{1}{4}p\right)^{\frac{5}{4}} e^{-\frac{27}{4}},$$

which for p large is $O(p^{\frac{5}{4}})$.

It follows that for small values of ω the maximum value of

$$I_\lambda(0) = \frac{2\omega}{1+\omega} K T_0{}^5 f(\alpha, \omega^{-1})$$

occurs near $\alpha_{\max} = 6\omega^{-\frac{1}{4}}$ and in magnitude is of the order of $\omega^{-\frac{1}{4}}$. Thus as $\omega \rightarrow 0$ we have the result stated. More precisely, for a given net flux, as the ratio of scattering coefficient to absorption coefficient tends to infinity $\lambda_{\max}$ tends to zero as the inverse eighth root of this ratio, and the intensity at $\lambda_{\max}$ tends to infinity directly as the eighth root. To preserve the area constant the slope of the curve on either side of the peak must become steeper, and the energy is all ultimately concentrated in a narrow range near the maximum.

Concentration of energy in the neighbourhood of the maximum is the dominant feature of the solar spectrum, but it is difficult to see any field for a plausible application of the foregoing result ; among other reasons, the steepness is not accompanied with sufficient displacement of the peak towards the blue.

* Paper 2, equations (29) and (33), p. 387.

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§12. *Mixture of Scattering and Absorption : the Effect on the Law of Darkening.*

It is not easy to decide *a priori* in what way the law of darkening will be modified by the presence of absorbing material in a scattering atmosphere—whether, for example, a small trace of absorbing material may not be sufficient to "colour" the radiation in accordance with the lower temperatures ruling in the scattering atmosphere, and so give an absorption law of darkening different for different colours rather than a scattering law. We proceed to prove that this is not the case.

The full equations, of which (39) and (40) are approximate forms, are

$$\cos\theta\frac{d\mathrm{I}_\lambda}{d\tau} = n_\lambda\mathrm{I}_\lambda - n_\lambda\xi_\lambda\mathrm{B}_\lambda - \tfrac{1}{2}(1-\xi_\lambda)\,n_\lambda\left[\int_0^{\frac{1}{2}\pi}\mathrm{I}_\lambda\sin\theta\,d\theta + \int_0^{\frac{1}{2}\pi}\mathrm{I}'_\lambda\sin\psi\,d\psi\right], \quad (63)$$

and a similar one for I'_λ. To solve them we require first an evaluation of the quantity in square brackets. For this purpose we adopt the approximate value of $\mathrm{I}_\lambda + \mathrm{I}'_\lambda$ given by (57), with 2τ written in place of t. Equation (63) then gives for the emergent radiation

$$\mathrm{I}_\lambda(0,\theta) = \int_0^\infty [\xi_\lambda\mathrm{B}_\lambda + \tfrac{1}{2}(1-\xi_\lambda)(\mathrm{I}_\lambda+\mathrm{I}'_\lambda)]\,e^{-n_\lambda\tau\sec\theta}\,n_\lambda\sec\theta\,d\tau. \quad . \quad . \quad . \quad (64)$$

If radiative equilibrium held for each separate wave-length, $\mathrm{I}_\lambda + \mathrm{I}'_\lambda$ would equal $2\mathrm{B}_\lambda$, and (64) would reduce to (14); (57) shows this is not the case. Using (57) with the value of P inserted and removing repeated integrations, we find, after somewhat laborious algebra,

$$\mathrm{I}_\lambda(0,\theta) = \frac{1+2\cos\theta}{1-4\omega_\lambda^2\cos^2\theta}\,\omega_\lambda(1-\omega_\lambda)\int_0^\infty \mathrm{B}_\lambda e^{-2n_\lambda\omega_\lambda\tau}\,2n_\lambda\omega_\lambda\,d\tau$$
$$-\frac{\omega_\lambda^2(4\cos^2\theta-1)}{1-4\omega_\lambda^2\cos^2\theta}\int_0^\infty \mathrm{B}_\lambda e^{-n_\lambda\tau\sec\theta}\,n_\lambda\sec\theta\,d\tau\,. \quad . \quad (65)$$

This is the general (approximate*) expression for the distribution and amount of the emergent radiation for combined absorption and scattering. By integration with respect to λ it may be verified that, if $n_\lambda = 1$ and $\omega_\lambda = \omega$, then the total radiation in direction θ is $\frac{1}{2}\mathrm{F}(1+2\cos\theta)$. For $\cos\theta = (2\omega_\lambda)^{-1}$ the expression given by (65) is indeterminate. The limit, however, as $\cos\theta \to (2\omega_\lambda)^{-1}$ is finite, and is found to be

$$\int_0^\infty \mathrm{B}_\lambda e^{-2n_\lambda\omega_\lambda\tau}\,2n_\lambda\omega_\lambda\,d\tau + (1-\omega_\lambda^2)\int_0^\infty\left(n_\lambda\omega_\lambda\tau - \frac{2+\omega_\lambda}{2+2\omega_\lambda}\right)\mathrm{B}_\lambda e^{-2n_\lambda\omega_\lambda\tau}\,2n_\lambda\omega_\lambda\,d\tau.$$

When $n_\lambda = 1$ and ω_λ is nearly unity, the second term here reduces to (49); the reason for this is easily seen.

Consider now the law of darkening given by (65). When there is no scattering, $\omega_\lambda = 1$ and (65) reduces to (14), the absorption law. When scattering is large compared with absorption, ω_λ is small and the first term in (65) is the dominant one; it gives a

* Analogous to the Schwarzschild-Schuster approximation.

law of darkening proportional to $1 + 2 \cos \theta$, independent of λ. For intermediate values of the scattering the law of darkening is also intermediate.

To obtain an estimate of the effect of a given amount of scattering in modifying the absorption law we proceed as follows. Assume that $\int_0^\infty B_\lambda e^{-n_\lambda \sec \theta \tau} n_\lambda \sec d\tau$ follows a cosine law approximately, say,

$$\int_0^\infty B_\lambda e^{-n_\lambda \tau \sec \theta} n_\lambda \sec \theta \, d\tau = C(1-u+u \cos \theta);$$

then

$$\int_0^\infty B_\lambda e^{-2n_\lambda \omega_\lambda \tau} 2n_\lambda \omega_\lambda \, d\tau = C\left(1-u+\frac{u}{2\omega_\lambda}\right),$$

provided $\omega_\lambda \geqslant \frac{1}{2}$, so that we are interpolating only. We find then from (65)

$$I_\lambda(0, \theta) = \frac{C(1+2\cos\theta)}{1+2\omega_\lambda \cos\theta}[(1-u+u\cos\theta)-(1-\omega_\lambda)(1-\tfrac{3}{2}u+u\cos\theta)]. \quad . \quad (66)$$

Now suppose the scattering is small, so that ω_λ is nearly unity. Put

$$\xi_\lambda = 1-\eta_\lambda, \qquad \omega_\lambda = 1-\tfrac{1}{2}\eta_\lambda,$$

where $\eta_\lambda = s_\lambda/(s_\lambda+k_\lambda)$ and is small. Then to the first order

$$I_\lambda(0, \theta) = C\left[1-u+u\cos\theta-\frac{\frac{1}{2}\eta_\lambda(1-\frac{3}{2}u)}{1+2\cos\theta}\right].$$

This formula indicates explicitly that when $u > \frac{2}{3}$ the scattering diminishes the darkening, and when $u < \frac{2}{3}$ it increases it. It thus always tends to make the darkening approach $u = \frac{2}{3}$. The mean coefficient of darkening $\bar{u}$ (see footnote, §4) is found to be given by

$$\bar{u} = u+\tfrac{1}{4}\eta_\lambda(\tfrac{2}{3}-u)(u+6-\tfrac{9}{2}\log 3)$$
$$= u+\tfrac{1}{4}\eta_\lambda(\tfrac{2}{3}-u)(u+1{\cdot}056). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

The following short table is calculated from this formula for $\eta_\lambda = \frac{1}{2}$; though this value is probably too large to be strictly permissible in (67).

u.	$\bar{u}$.	u.	$\bar{u}$.
1·0	0·91	0·6	0·61
0·9	0·84	0·5	0·53
0·8	0·77	0·4	0·44
0·75	0·73	0·3	0·36
0·67	0·67		

It must be remembered that (67) gives the effect of scattering in modifying the darkening, over and above its effect in modifying n_λ; we may term n_λ the "coefficient of obstruction."

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Lastly, we will calculate the ratio of the mean intensity in λ over the disc to the mean intensity in the absence of scattering, for the same coefficient of obstruction. It is found to be, for η_λ small,

$$1-\eta_\lambda\left(\tfrac{1}{2}-\tfrac{1}{4}\log 3\right)\frac{1-\frac{3}{2}u}{1-\frac{1}{3}u} = 1-0\cdot 225\eta_\lambda\frac{1-\frac{3}{2}u}{1-\frac{1}{3}u}.$$

If, therefore, λ is such that $u > \frac{2}{3}$, the mean intensity is increased by a small amount of scattering, if $u < \frac{2}{3}$ diminished. This result is allied to that deduced from equation (49), §11.

The general effect of this section is to confirm the difference between scattering and absorption in their influence on the darkening.

§13. *Concluding Remarks on Scattering and the Solar Darkening.*

If a scattering atmosphere forms part of the outer layers of the sun, or exists outside the visible disc, it may be supposed *a priori* to be optically either thin or thick, and geometrically either thin or thick, with any combination between these. Now the foregoing investigation, treating as it has done of spherical strata of matter of radius so large that in any particular region they may be considered plane, applies to a layer or atmosphere of thickness *small* compared with the radius of the sun. The results appear to rule out the possibility of the existence on the sun of a shallow scattering layer of this type, which should be optically thick enough to give rise to the observed distribution of energy in the continuous spectrum, and yet optically thin enough to give the observed law of darkening in the different wave-lengths. In so far as the gases constituting the chromosphere and reversing layer act as purely scattering media, their total optical thickness must be negligible; and in so far as the outer layers of the visible sun scatter the photospheric light which they are in part contributing, the ratio s/k must be so small as to be negligible.

We can dismiss the possibility that a *large* thickness of the visible sun might consist of purely or even largely scattering material. There remains the possibility that outside the visible disc there may be an extensive scattering atmosphere of geometrical thickness comparable with the sun's radius and of adequate optical thickness. Though the mathematical embodiment of this has not been here attempted*—a difficult problem, since curvature would have to be taken into account—it appears that the objections mentioned at the end of §7 would hold, *i.e.* if the optical thickness were sufficient to have the necessary effect on the continuous spectrum, the details of the distribution over the disc would be altered, and further there should be a considerable intensity outside the visible disc. This shows that the corona does not play the required part, and that, though the corona conspicuously illustrates some of the phenomena associated

* *Cf.* SCHUSTER, "Polarisation of the Solar Corona," 'M.N., R.A.S.,' 40, p. 35 (1879).

with scattering, its total optical thickness must be very small. It does not seem possible to suggest a form of scattering atmosphere which will leave undisturbed, or only slightly disturbed, the law of darkening based on radiative equilibrium without scattering whilst providing for the observed peculiarities in the continuous spectrum.

§14. *Summary and Conclusions.*

It is shown that if T is the effective temperature of a star in radiative equilibrium the law of darkening in wave-length λ is a function of λT (or λ/λ_{max}) only. The darkening for given T is less the bigger is λ, and thus for given λ the darkening is less the bigger is T. For large values of λT (or λ/λ_{max}) the darkening tends to a limiting value, for which the limb-centre ratio is about $0 \cdot 8$, and the darkening at the limb can never be less than this. For small values of λT (λ/λ_{max} a small fraction) the darkening at the limb tends to become complete. If the law of darkening in integrated light is given, and if the matter composing the star is assumed to be gray, it is shown that the temperature distribution in the outer layers is determinate and calculable as a function of the optical depth. This temperature distribution will not in general correspond to radiative equilibrium, and the degree of convection of energy necessary to sustain it is calculable; but for stars which follow a strict cosine law of darkening the deduced temperature distribution is not physically possible unless the coefficient of darkening lies between $\frac{3}{5}$ and $\frac{1}{2}$. Hence on the conditions stated the coefficient of darkening of an actual star must lie between these limits.

The law of darkening for non-gray matter in radiative equilibrium is determined; it is shown that where the curve giving the energy in the continuous spectrum lies above the corresponding black body curve of equal area, the darkening should be greater than the normal and conversely.

The influence of various types of scattering atmospheres is then analysed. It is emphasised that for an optically thick scattering atmosphere the law of darkening is independent of the scattering coefficient and so independent of wave-length; the law is identical with that for the integrated radiation for gray material in radiative equilibrium; further, the law is independent of the darkening of the background. An optically thin scattering atmosphere has two effects: if the background is provided by radiating material in radiative equilibrium the scattering layer acts as a blanket, keeping the background warmer than it would otherwise be, and so leading to a law of darkening for the background corresponding to a higher temperature; the scattering layer then modifies this law in the direction of reducing the coefficient of darkening in each wave-length to the same constant quantity. Numerical examples are given.

Lastly, the law of darkening is determined for matter which both scatters and absorbs; a formula is found which shows that the law approaches the absorption law or the scattering law according to the relative predominance of the coefficients of absorption and scattering. In the course of this latter work, it is found necessary to determine the effect of scattering on the continuous spectrum; a formula is obtained which embodies SCHUSTER's results combined with the condition of radiative equilibrium, and some curious hypothetical special cases are discussed.

The theory is applied to the case of the observed darkening on the sun. This darkening is slightly less than that corresponding to the temperature given by the solar constant. But it does not show such correlation with the observed form of the spectral energy curve as would be anticipated if the distortion of the energy curve were due to a dependence of absorption on wave-length in the layers contributing the radiation.

The inference is that the form of the energy curve cannot be attributed to any such selective absorption and emission in the radiating layers.

Again, it is shown that the marked dependence of darkening on wave-length precludes the possibility of an optically thick scattering layer as the cause of the distortion. A sufficiently thin scattering layer would indeed raise the temperature of the radiating layers and so bring the darkening of the photospheric radiation into accordance with observation; but in the first place the increase of temperature required for this is much too small to account for the magnitude of the maximum intensity in the spectrum; and in the second place even a thin scattering atmosphere should alter the darkening of the photospheric radiation in a manner not borne out by observation.

It is deduced with some confidence that scattering plays no appreciable part in the solar phenomena here considered; the ratio of the coefficient of scattering to the coefficient of absorption in the sun's outer layers must be negligible, and the bulk of the emergent radiation is not scattered light.

The paradox, that the darkening of the sun's disc is very closely the same as it would be if the sun were a black body whilst the spectrum itself indicates a considerable departure from black body conditions, remains unexplained. The hypothesis of non-black photospheric emission and the hypothesis of a scattering atmosphere are those which it is natural to make, but they prove to be inadequate. Whether or no the solution is to be found in processes occurring in the thin layers producing the Fraunhofer spectrum remains for future investigation.

It should be stated explicitly that this discussion refers only to the mean condition of the undisturbed solar surface.

Professor NEWALL first suggested to me the desirability of investigating further the theory of stellar atmospheres, and my best thanks are due to him for his kind criticism and discussion of many points.

APPENDIX I.—*The function* $f(a, p)$.

In the numerical applications of the theory it is necessary to have values of the function $f(a, p)$ defined by equation (3A), p. 206. Unfortunately, this function is very laborious to evaluate. Some of the difficulties have been referred to in the Appendix to Paper 2, where the concern was chiefly with the case $p = 1$. For large values of a the asymptotic formula can be employed for any value of p, but for small values of

α the calculation by means of BERNOULLI's expansion is very tedious, since a fresh set of definite integrals are required for each value of p, unless recourse is had to incomplete Γ-functions. Accordingly, for small values of α, I have availed myself of a short table given by LINDBLAD.* Partly by numerical integration (SIMPSON's rule), and partly by a graphical method, he determined a set of values of a function K (θ, ϵ), which in our notation is given by

$$\mathrm{K}\,(\theta,\,\epsilon) = \alpha^{-5} f\,(\alpha,\,p)\,[e^{\alpha\,(1+p)^{-\frac{1}{4}}}-1]$$

where

$$\epsilon = p, \qquad \theta = \alpha\,(1+p)^{-\frac{1}{4}}.$$

From this, $f\,(\alpha, p)$ may be deduced, for even values of p but broken values of α. The values thus derived are the earlier ones given in the tables below; the later ones were found from the asymptotic formula. It is satisfactory to be able to record that LINDBLAD's values, obtained by direct numerical integration, are in good agreement with those determined analytically from the Bernoulli series or the asymptotic formula; *e.g.* for $p = 1$, $\alpha = 6{\cdot}5$, $\theta = 5{\cdot}4658$, K $(\theta, \epsilon) = 1{\cdot}0148$, whence $f\,(\alpha, p) = 50{\cdot}00$, whilst the value found from the asymptotic formula (which is admittedly only approximate near $\alpha = 6{\cdot}5$) is 49·91.

In the calculation of Table I., pp. 209, 210, more extensive values of $f\,(\alpha, p)$ were required; these were obtained from those tabulated below by graphical interpolation.

$p = 0.$

α.	α^{-1}.	$f\,(\alpha, p)$.	α.	α^{-1}.	$f\,(\alpha, p)$.
1·0	1·0000	0·582	5·3	0·1887	20·98
1·4	0·7143	1·760	6·0	0·1667	19·32
1·7	0·5882	3·174	7·0	0·1429	15·34
2·0	0·5000	5·01	8·0	0·1250	11·00
2·5	0·4000	8·73	9·0	0·1111	7·29
3·0	0·3333	12·73	10·0	0·1000	4·54
3·5	0·2857	16·35	11·0	0·0909	2·690
4·0	0·2500	19·10	12·0	0·0833	1·529
4·5	0·2222	20·73	16·0	0·0625	0·118
5·0	0·2000	21·20			

$p = 0{\cdot}5.$

α.	α^{-1}.	$f\,(\alpha, p)$.	α.	α^{-1}.	$f\,(\alpha, p)$.
1·1057	0·9044	0·954	8·0	0·1250	24·87
2·2134	0·4518	8·18	10·0	0·1000	13·26
3·3200	0·3012	20·80	12·0	0·0833	5·90
4·4267	0·2259	31·34	16·0	0·0625	0·854
5·5334	0·1807	35·51			
6·6401	0·1506	32·58			

* *Loc. cit.*, p. 33.

$p = 1.$

[A table for $p = 1$ has been given in Paper 2, p. 380 (the function there called $\phi_1(\alpha)$). For completeness, a selection of these values is given, besides additional values from Lindblad.]

α.	α^{-1}.	$f(\alpha, p)$.	α.	α^{-1}.	$f(\alpha, p)$.
1·1892	0·8409	1·346	5·9461	0·1682	50·35
2·3784	0·4205	11·51	7·1353	0·1401	47·48
3·5676	0·2803	29·26	9·5137	0·1051	30·16
4·7568	0·2102	44·36			
1·0	1·0000	0·739	6·5	0·1538	49·91
1·5	0·6667	2·912	7·0	0·1429	48·17
2·0	0·5000	7·076	8·0	0·1250	42·04
2·5	0·4000	13·12	9·0	0·1111	34·24
3·0	0·3333	20·46	10·0	0·1000	26·45
3·5	0·2857	28·22	11·0	0·0909	19·62
4·0	0·2500	35·55	12·0	0·0833	14·08
4·5	0·2222	41·76	14·0	0·0714	6·67
5·0	0·2000	46·38	16·0	0·0625	3·00
5·5	0·1818	49·25	20·0	0·0500	0·517
6·0	0·1667	50·36			

$p = 1 \cdot 5.$

α.	α^{-1}.	$f(\alpha, p)$.	α.	α^{-1}.	$f(\alpha, p)$.
1·2574	0·7953	1·757	6·6	0·1515	66·71
2·5149	0·3976	14·97	6·7	0·1493	66·61
3·7723	0·2651	38·10	9·0	0·1111	52·06
5·0297	0·1988	58·12	10·0	0·1000	42·53
6·2872	0·1591	66·54	11·0	0·0909	33·36
7·5446	0·1325	63·66	13·0	0·0769	18·73
6·3	0·1587	66·53	16·0	0·0625	6·75
6·5	0·1539	66·75	20·0	0·0500	1·463

$p = 2 \cdot 0.$

α.	α^{-1}.	$f(\alpha, p)$.	α.	α^{-1}.	$f(\alpha, p)$.
1·3161	0·7598	2·179	10·5286	0·0950	54·70
2·6321	0·3799	18·44	7·0	0·1429	83·21
3·9482	0·2533	47·32	7·899	0·1266	80·41
5·2643	0·1900	72·50	9·0	0·1111	71·39
6·5804	0·1520	83·57	12·0	0·0833	41·96
7·8964	0·1266	80·88	16·0	0·0625	12·19

APPENDIX II.—*Radiation Constants. The Sun's Energy Curve on an Absolute Scale.*

The intensity of black radiation, B, is given by the formula $B = bT^4$. The constant b is connected with STEFAN'S constant σ, PLANCK'S constant h, the gas-constant R, the velocity of light c and the constant K defined in §2 by the relations

$$b = \frac{\sigma}{\pi} = \frac{\pi^4}{15}\left(\frac{hc}{R}\right) K = \frac{2\pi^4}{15} hc^2 \left(\frac{R}{hc}\right)^4.$$

If we take $c = 2\cdot999 \times 10^{10}$ cms. sec.$^{-1}$, and adopt MILLIKAN'S value for the charge on an electron, $e = 4\cdot774 \times 10^{-10}$ E.S.U., and the value 96,468 coulombs for FARADAY'S constant, then AVOGADRO'S constant N, the number of molecules in 1 cm.^{3} of gas at 0° C. (273·1° abs.) and 1 atmosphere ($1\cdot01323 \times 10^6$ dynes) comes out as $2\cdot707 \times 10^{19}$, and the value of R is $1\cdot3706 \times 10^{-16}$. (The standard density of oxygen is taken as 1·4290 grams/litre.) The value of h deduced from RYDBERG'S constant is $6\cdot547 \times 10^{-27}$ gram. cm.^{2} sec.$^{-1}$, and is a satisfactory value to adopt. We then find for the various radiation constants:

$$hc/R = 1\cdot4325, \qquad \lambda_{max}\, T = \frac{1}{4\cdot9651} \cdot \frac{hc}{R} = 0\cdot28852,$$

$$b = 1\cdot816 \times 10^{-5}, \qquad \sigma = 5\cdot705 \times 10^{-5},$$

$$K = 1\cdot952 \times 10^{-6}.$$

(This value of σ is very close to that found directly by COBLENTZ.*) Further, it is easy to show that the ordinate of maximum energy in the normal spectrum has the value

$$(B_\lambda)_{max} = KT^5 x^4 (5-x) = 21\cdot210\, KT^5, \qquad (68)$$

where x is the constant 4·9651.

Now let πF_λ be the total radiation from the sun, in λ, per unit area. It has been shown in the *Note*, § 4 (p. 212), that F_λ is equal to the mean intensity over the apparent disc, in λ. Let T_1 be the effective temperature, as determined by the total radiation, and let T_0 be defined by $2\,T_0^4 = T_1^4$. Then

$$F = \int_0^\infty F_\lambda d\lambda = bT_1^4 = \frac{2\pi^4}{15}\frac{hc}{R} KT_0^4.$$

Further, let $I_\lambda(\theta)$ be the intensity at an angle θ with the normal, which is the same as the observed intensity at an angular distance θ from the centre of the disc; and let $\bar{u}$ be the observed mean coefficient of darkening. Then, as shown in the *Note* just referred to,

$$I(0) = \int_0^\infty I_\lambda(0)\, d\lambda = \frac{F}{1-\frac{1}{3}\bar{u}}.$$

* 'Bull. Bur. Stand.,' 12, p. 553 (1915).

Now introduce two quantities $[F_\lambda]$, $[I_\lambda(\theta)]$, defined by

$$F_\lambda = KT_0^5[F_\lambda], \qquad I_\lambda(\theta) = KT_0^5[I_\lambda(\theta)] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (69)$$

The quantities just defined are comparable with the function $f(\alpha, p)$, which occurs in theoretical investigations, and their employment is the most useful way of recording the sun's spectral energy curve in absolute units. But it must be noted that they have been carefully defined so as to be independent of any theory. They enjoy the properties.

$$\int_0^\infty [F_\lambda]\,d\lambda = \frac{2\pi^4}{15}\cdot\frac{hc}{RT_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (70)$$

$$\int_0^\infty [I_\lambda(0)]\,d\lambda = \frac{2\pi^4}{15}\cdot\frac{hc}{RT_0}\cdot\frac{1}{1-\frac{1}{3}\bar{u}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (71)$$

where all the quantities are expressed in absolute units and in particular λ is expressed in cms.

Below are given the values of $[F_\lambda]$ and $[I_\lambda(0)]$, according to ABBOT, FOWLE and ALDRICH, and the values of $[F_\lambda]$, according to WILSING; I have determined them from the published results by numerical integration. The Smithsonian observers give two very valuable tables of intensities; but both are on an arbitrary scale, and, unfortunately, on different arbitrary scales, and for different wave-lengths. One gives ordinates of the mean energy curve*; the other† gives the energy curves for a series of points on the disc. The former was plotted, integrated numerically, and then multiplied by such a factor as would make (70) valid. If, in accordance with a solar constant of 1·932 cal. cm.$^{-2}$ min.$^{-1}$, we take

$$T_1 = 5740, \qquad T_0 = 4830,$$

the factor is found to be 0·01110. But $[I_\lambda(0)]$ cannot be determined so simply; for $\bar{u}$ is required, and this involves integrating the energy curves for the separate places on the disc and then integrating the results over the surface of the disc. This has been done, as already stated in §4. The result is that

$$1-\tfrac{1}{3}\bar{u} = 0{\cdot}8205.$$

It is then found that the central intensities in ABBOT's Table 56 require to be multiplied by the factor 0·16705, in order to satisfy (71); this factor can accordingly be used all over the disc, and (69) then gives $I_\lambda(0)$ in absolute units. For comparison, the values of $[F_\lambda]$ have been read off for the wave-lengths used for $[I_\lambda(0)]$. For brevity only the central intensities are given; the others can be found from ABBOT's table.

In making the integrations allowances have to be made for the unascertained portions of the curves in the ultra-violet and infra-red. These were calculated by fitting the

* 'Annals,' 3, p. 197; or 'Astrophys. Journ.,' 37, p. 197 (1911).

† 'Annals,' 3, p. 159.

appropriate arc of a Planck curve on to the plotted curve. The areas added are quite appreciable, but the actual doing of the work convinces one that the corrections thus made cannot be far wrong.

WILSING'S* curve, derived by combining photographic and bolometric observations, refers to the mean radiation only ; hence only $[F_\lambda]$ can be deduced. The antilogarithms corresponding to his tabular logarithms require to be multiplied by the factor 2·277.

λ (A.U.).	$[F_\lambda]$ (ABBOT).
3000	5·98
3250	14·11
3500	29·79
3750	38·39
3900	40·12
4200	58·29
4300	59·06
4500	66·90
4700	69·26
5000	67·29
5500	62·42
6000	55·97
7000	40·45
8000	29·58
10,000	18·39
13,000	9·97
16,000	5·91
20,000	2·74
25,000	0·48
30,000	0·16

λ (A.U.).	$[I_\lambda(0)]$ (ABBOT).	$[F_\lambda]$.
3230	24·05	13·3
3860	56·46	41·4
4330	76·17	61·4
4560	86·03	68·4
4810	85·36	69·0
5010	81·69	67·3
5340	77·34	64·2
6040	66·65	55·0
6700	55·63	44·3
6990	51·28	40·5
8660	29·07	25·5
10,310	18·54	17·2
12,250	12·96	11·4
16,550	6·60	5·3
20,970	2·34	2·3

* 'Pub. Astrophys. Obs. Potsdam,' vol. 23, No. 72, p. 28 (1917).

λ (A.U.).	$[F_\lambda]$ (WILSING).
4000	45·7
4200	50·7
4400	55·5
4600	59·9
4800	63·8
5000	63·5
5200	60·7
5400	57·3
5600	54·5
5800	52·5
6000	50·8
6500	46·6
7000	42·3
7500	38·0
8000	33·8
9000	26·8
10,000	21·2
11,000	16·9
12,000	13·4
13,000	10·8
14,000	8·8
15,000	7·2
16,000	5·9
17,000	4·9
18,000	4·0
19,000	3·3
20,000	2·8
21,280	2·2

From these tables we can make an interesting deduction: we can readily calculate the temperature indicated by the magnitude of maximum ordinate in the spectrum. For combining (68) with the definition of $[F_\lambda]$ we have, if T_{max} is the temperature in question,

$$T_0^5 [F_\lambda]_{max} = 21 \cdot 210 \; T_{max}^5.$$

The greatest value of $[F_\lambda]$ occurring amongst ABBOT's numbers is seen to be 69·26, and it is sufficient to take this as the maximum. This gives

$$T_{max}/T_0 = 1 \cdot 267, \qquad T_{max} = 6120°.$$

This should be compared with that deducible from WIEN's law; according to ABBOT, $\lambda_{max} = 4700$, which, with our radiation constants, gives a temperature of 6130°. WILSING's maximum ordinate gives a temperature of 6020°, and his λ_{max} a temperature of about 5900°.

I am not aware that an estimate of the temperature indicated by the *magnitude* of the maximum ordinate has been previously given, though it must have been present in the minds of many writers. The lowness of this temperature (less than 400° above that indicated by the solar constant) and its agreement with the temperature deduced from λ_{max}, are interesting and perhaps surprising.

Critical Note.

Papers dealing with the absorption and scattering of light in the solar atmosphere and with the consequent darkening of the disc may be divided into three classes: (1) those which contemplate absorption but not radiation, and which ignore the resulting accumulation of energy; (2) those which take into account the possibility of radiation from the absorbing layer, but fail to take care that the emitted and absorbed radiations balance; (3) those which take full account of the equilibrium of radiation. This order is roughly the order in which the different types of theory were put forward; for example (and other instances could be given), we have (1) Very's paper ('Astrophys. Journ.' pp. 16, 73, 1902), (2) Schuster's paper ('Astrophys. Journ.,' 16, p. 320, 1902), (3) Schwarzschild's paper ('Gött. Nach.,' 1906, p. 41). Very assumed a non-radiating absorbing layer in front of a radiating photosphere, and deduced "transmission coefficients" for this layer, from consideration of its greater effective thickness near the limb, by the usual Bouguer-Lambert secant formula. Schuster accounted for the observed intensity distribution by taking into account the effect of the radiation emitted by the absorbing layer, assuming this to be at a constant temperature; but he had to use the observed intensity at three places on the disc to determine the three unknowns—the intensity of the photospheric radiation and the temperature and optical thickness of the absorbing layer; his formula was thus in reality an interpolation formula. Schwarzschild in effect removed the empiricism from Schuster's formula by balancing the radiation from the layer against the loss of photospheric radiation. But in this treatment the distinction between photosphere and absorbing layer disappears; each shades continuously into the other; the absorption coefficient of the material no longer appears explicitly, and the law of darkening of the integrated radiation is free from all arbitrary constants whatever. Further, as has been shown, it is a straightforward deduction from Schwarzschild's theory that for the darkening in the separate colours a single quantity alone requires to be determined—the effective temperature.

It is remarkable that a theory so simple, so logical and showing such general accordance with the facts should have sprung to birth with such completeness. Its beauty is best realised when we contrast it with a paper such as that of Öpik ('Ast. Nach.,' 198, p. 49, 1914), which, clearly written in ignorance of Schwarzschild's work, gropes complicatedly in the same direction.

But the implications of Schwarzschild's paper have not always been fully realised. In some interesting "remarks on the hypotheses used for explaining the distribution of the radiating power of the solar disc," Julius* stated explicitly the objection to the then existing theories. ". . . . The fundamental idea that a considerable portion of the photospheric radiation should be absorbed by a thin atmosphere encounters a difficulty of greater importance still. This point, I think, has also first been raised by Schmidt. What becomes of the absorbed energy accumulating in the atmosphere?

* 'Astrophys. Journ.,' 23, p. 322 (1906).

According to SCHUSTER [*loc. cit.*], the atmosphere transmits largely one-third of the radiation emitted by the photosphere; so it stops almost two-thirds, and only a small fraction of this absorbed energy leaves the sun in the form of radiation emitted by the atmosphere itself . . . As long as it has not been shown how the solar atmosphere may get rid of that immense quantity of energy continually supplied and never radiated, similar considerations will remain very unsatisfactory." The problem thus raised was solved by SCHWARZSCHILD almost at the same moment. Yet so late as 1916 there appeared a paper by BISCOE,* which asserted that "the gaseous strata lying above the photosphere, on account of their relatively small density, and hence also on account of small temperature, can yield only a very inappreciable radiation in comparison with the photosphere; they reveal themselves, by a selective and general absorption of the photospheric radiation, as the solar atmosphere"; and which objected to SCHWARZSCHILD'S theory partly on the ground that "the equilibrium of radiation leads to an incomprehensible atmosphere which reaches into infinity with a very high constant temperature at its extreme strata." BISCOE proceeded to determine transmission coefficients and refractive indices for the solar atmosphere, and to deduce a photospheric temperature of $7300° \pm 100°$—a value which has been freely quoted in a recent series of papers.†

In view of this and other papers, it may be as well to point out, in the first place, that SCHWARZSCHILD'S theory of radiative equilibrium does not lead to an atmosphere reaching to infinity with a high constant temperature. SCHWARZSCHILD in his paper calculated the densities implied by the temperature distribution for radiative equilibrium, and showed that the density fell off with exceeding rapidity‡ outside the limb—that the theory (in common with most theories) appeared to demand an even sharper edge to the sun than is actually observed. But, more than this, the constant boundary temperature given on the simple theory of radiative equilibrium applies only to regions so near the surface that curvature may be neglected. For simplicity in the mathematics the theory is exposed as if for a distribution of matter stratified in parallel planes; but it is obvious without calculation that as soon as the effect of curvature begins to be appreciable the fourth power of the temperature (for gray material) will fall off as the inverse square of the distance, or the temperature as the inverse square root of the distance.§ If any direct evidence is required of the reality of the SCHWARZSCHILD boundary temperature at small distances, one has only to consider the earth's stratosphere. The vertical temperature gradient over any place is small or zero, and the air is in radiative equilibrium under the influence of the earth's radiation (the direct absorption of sunlight being almost negligible); there is a net

* 'Astrophys. Journ.,' 43, p. 197 (1916). See also criticisms by ABBOT and FOWLE in the same journal, 44, p. 39 (1916), and BISCOE'S reply, 46, p. 355 (1917).

† SAHA, 'Phil. Mag.,' 40, p. 472 (October, 1920), etc.

‡ See also BRUNT, 'M.N., R.A.S.,' 73, p. 571 (1913).

§ MILNE, 'M.N., R.A.S.,' 82, p. 368, 1922.

outward flux equal to the rate of absorption of solar energy by the earth and lower atmosphere; and the temperature calculated by the $\sqrt[4]{2}$ rule is in good agreement with the actual temperature, as was first shown by HUMPHREYS.*

In the second place, it should be pointed out that any attempt to calculate directly the "transmission coefficients of the solar atmosphere" by crude methods must in the light of SCHWARZSCHILD'S work be considered hopeless. For the theory provides a definite relation between temperature and optical depth involving only one constant, the effective temperature. Suppose now, ignoring this theory, we arbitrarily select a certain temperature, and name it the photospheric temperature, and name the unknown depth at which it occurs the photospheric depth; this depth will be described by some unknown transmission coefficient, to be determined. If, taking account of absorption and emission, we proceed to calculate the transmission coefficient by comparing the calculated brightness distribution with the observed, we shall simply recover the optical depth predicted by SCHWARZSCHILD'S theory. We may, if we choose, remove the arbitrariness of an assigned photospheric temperature by imposing some other condition, *e.g.* one based on refraction; but this again merely has the effect of selecting some particular layer, and the transmission coefficients are again meaningless. The point depends on the fact that the law of darkening based on radiative equilibrium *very nearly* fits the observed darkening with an absorption coefficient the same for all wave-lengths and arbitrary in value; hence the true values of the absorption coefficients can only be determined, if at all, from the discrepancies. If we abandon purely radiative equilibrium, and ensure thermal equilibrium by allowing some rôle to convection, the case is stronger still; any darkening law then implies not a temperature and transmission coefficients, but a temperature distribution as a function of optical depth—again independent of the actual value of the absorption coefficient (see §5 above).

The misunderstandings on the subject of solar absorbing atmospheres have been due in some measure to a failure to realise that the terrestrial atmosphere offers no analogy. Here the weakening of sunlight by absorption can be considered without enquiry as to the absorbed radiation, because this becomes eventually converted into low-temperature radiation in an entirely different region of the spectrum. The radiation the atmosphere has to deal with is not sufficient in amount to raise its temperature to one comparable with that of the source of the radiation. The contrary is the case on the sun; the theory of radiative equilibrium shows that the absorbing layers there cannot dispose of their absorbed energy as low-temperature radiation, but must rise to a temperature comparable with that of the emitting layers behind them.

* 'Astrophys. Journ.,' 29, p. 14 (1909).